Environmental policy, education and growth: A reappraisal when lifetime is finite

Xavier Pautrel

Document de Travail
Working Paper

Lemna

EA 4272

Environmental policy, education and growth: A reappraisal when lifetime is finite

Xavier PAUTREL (*)

2009/27

(*) LEMNA, Université de Nantes

Laboratoire d'Economie et de Management Nantes-Atlantique
Université de Nantes
Chemin de la Censive du Tertre – BP 52231
44322 Nantes cedex 3 – France
www.univ-nantes.fr/iemn-iae/recherche
Tél. +33 (0)2 40 14 17 19 – Fax +33 (0)2 40 14 17 49

INSTITUT d'ECONOMIE et de MANAGEMENT de NANTES - IAE

UNIVERSITÉ DE NANTES

Environmental policy, education and growth: A reappraisal when lifetime is finite

Xavier Pautrel*†

October 1, 2009

Abstract

This article demonstrates that when finite lifetime is introduced in a Lucas (1988) growth model where the source of pollution is physical capital, the environmental policy may enhance the growth rate of a market economy, while pollution does not influence educational activities, labor supply is not elastic and human capital does not enter the utility function. The result arises from the "*generational turnover effect*" due to finite lifetime. It remains valid under conditions when the education sector uses final output besides time to accumulate human capital. Nevertheless, it does no longer hold when the source of pollution is output.

Furthermore, this article demonstrates that ageing reduces the positive influence of the environmental policy when growth is driven by human capital accumulation à la Lucas (1988) and lifetime is finite. It also confirms for finite lifetime the result found by Vellinga (1999) with a single representative agent: environmental care does not influence optimal growth when utility is additive and pollution does not influence the ability of agents to be educated.

Keywords : Growth; Environment; Overlapping generations; Human capital;

*Nantes Atlantique Université, Laboratoire d'Économie et de Management de Nantes, Institut d'Économie et de Management de Nantes - IAE, Chemin de la Censive du Tertre, BP 81307, 44313 Nantes Cedex 3, France. *Mail*: `xavier.pautrel@univ-nantes.fr`

†I thank Fabien Tripier for a stimulating discussion about a previous version of this paper and Yan Rebillé for helpful comments. The usual disclaimer applies. This working paper is a revised version of the FEEM Working paper 216 (July 2008).

1 Introduction

While the link between environment and growth has already been extendedly investigated (see Xepapadeas, 2005; Brock and Taylor, 2005), conclusions about the influence of the environmental policy on economic growth remain open. The purpose of this article is to contribute to the debate, by re-examining the influence of the environmental policy on human capital based-growth when finite lifetime is taken into account. It demonstrates that finite lifetime introduces a new channel of transmission between the environment and economic performances based on the turnover of generations.

Most of the industrialized countries are now becoming knowledge- and education-based economies using more and more human capital instead of physical capital to produce. And education played a major role in the industrialization of the South-East Asian countries during 7os and 80s decades.[1] Nevertheless, few theoretical works investigate environmental issues in a framework where human capital is the engine of growth and economic prosperities.[2] A noteworthy exception is the seminal article by Gradus and Smulders (1993). In a model à la Lucas (1988) where pollution originates from physical capital, they demonstrate that the environment never influences the steady-state growth rate except when pollution affects education activities.[3] This result comes from the fact that the growth rate of consumption relies on the after-tax interest rate and the rate of time preferences and that the tax-rate is invariant with pollution tax in the steady-state when labor supply is inelastic. When the environmental taxation increases,

[1]See World Bank (1993) for empirical evidence of this role. Grimaud and Tournemaine (2007) point out this role to justify the need to investigate the link between environment and growth through the channel of education.

[2]Here, we do not deal with the major question of climate change and we do not integrate non-renewable resources in the analysis. See Schou (2000, 2002) and Grimaud and Rouge (2005, 2008) for authors who investigate environmental policy and growth in the presence of non-renewable resources and endogenous growth. Note also that there exists an important literature on the impact of environmental policy on growth, even if the contributions do not deal with human capital accumulation: on the double dividend, see Bovenberg and Smulders (1995, 1996), Bovenberg and de Mooij (1997); for contributions using OLG model, see Ono (2002, 2003)(discrete time model) or Bovenberg and Heijdra (1998, 2002) (continuous time Yaari-Blanchard model) amongst others.

[3]More precisely, they assume that pollution depreciates the stock of human capital. van Ewijk and van Wijnbergen (1995) consider that pollution reduces the ability to train .

the after-tax interest rate reduces and becomes lower than the returns to human capital. Consequently, the investment in physical capital drops in favor to human capital accumulation. Final production becomes more intensive in human capital and the allocation of human capital in production diminishes. This mechanism perpetuates until the after-tax interest rate backs to its initial value equal to the rate of returns in human capital accumulation. Because the aggregate consumption growth in the steady-state relies only on the after-tax interest rate, it is not modified by the higher pollution tax.

Assuming that labor supply is elastic and pollution originates from the stock of physical capital, Hettich (1998) finds a positive influence of the environmental policy on human capital accumulation, in a Lucas' setting. The increase in the environmental tax compels firms to increase their abatement activities at the expense of the household's consumption. To counteract this negative effect, households substitute leisure to education and the growth rate rises. Nevertheless, the author demonstrates that his result is very sensitive to the assumption about the source of pollution. When pollution originates final output rather than physical capital, the link between the environment and growth does not longer exist. By taxing output, a tighter environmental policy reduces both the returns to physical capital and the wage rate which contributes to the returns to education. The incentives of agents to invest more in education vanish.

More recently, Grimaud and Tournemaine (2007) demonstrate that a tighter environmental policy promotes growth, in a model combining R&D and human capital accumulation, where education directly enters the utility function as a consumption good and knowledge from R&D reduces the flow of pollution emissions. By increasing the price of the good whose production pollutes the higher tax rate reduces the relative cost of education and therefore incites agents to invest in human capital accumulation. Because education is the engine of growth, the growth rises in the steady-state. As highlighted by the authors, the key assumption is the introduction of the education as a consumption good in utility which lets the returns to education dependent from the environmental policy. When education does not influence utility, the returns to education is exogenous and therefore is not affected by the policy.

In the present article, we re-examine the link between the environmental policy and growth in a Lucas' setting, assuming just that lifetime is finite. We use a Yaari (1965)-Blanchard (1985) overlapping generations model where growth is driven by human capital accumulation à la Lucas (1988) and pollution arises from physical capital.[4] We study both the long-run balanced growth path and the transition.

Our results are fourfold. First, we demonstrate that when lifetime is finite and physical capital is the source of pollution, a tighter environmental policy enhances growth even if pollution does not affect educational activities, labor supply is inelastic and human capital does not enter the utility function. Indeed, finite lifetime introduces a turnover of generations which disconnects aggregate consumption growth to the interest rate. This effect appears because at each date a new generation is born and a cross-section of the existing population dies. Because new agents born without financial assets, their consumption is lower than the average consumption and therefore the "*generational turnover effect*" reduces the growth rate of the aggregate consumption. This *generational effect* rises with the probability to die because on one hand agents die at a higher frequency (that increases the generational turnover), and on the other hand the propensity to consume out of wealth increases due to the shorter horizon. Because the "*generational turnover effect*" leads the aggregate consumption growth to be higher than its initial level while the after-tax interest rate remains unchanged, it incites agents to invest more in human capital (the non-polluting factor) and therefore creates a growth-enhancing effect of the environmental tax.

We also demonstrate that, in the presence of finite lifetime, the ageing of the population (a

[4]Following Gradus and Smulders (1993) and many other authors, we model pollution as a flow that originates from production. It corresponds to pollutant emissions like, for example, untimely noise, or non-permanent volatile organic compounds produced by the industry or generated by industrial process (like the use of solvents in consumer and commercial products that corresponds to 28% of the total emissions of VOCs in Canada for 2000). Like (Hettich, 1998, p.292), one should make observe that the "flow" assumption (rather than the "stock" assumption) does not modify the qualitative results along the Balanced Growth Path.

lower probability to die) reduces the positive influence of the environmental policy on growth, for the aforementioned reasons.

Second, when we relax the assumption that only time is required to be educated and we assume that an education good is used as input in education activities, the "*generational turnover effect*" continues to operate and the environmental policy enhances growth, only if the part of the education good in human capital accumulation remains small. Otherwise, the crowding-out effect leads the environmental policy to be harmful to growth.

Third, when the source of pollution is final output rather than physical capital, we demonstrate that the environmental policy does not influence growth in the Lucas (1988) setting, because the wage rate (that influences the returns to education) is also affected. As a result, the substitution between physical capital and human capital is reduced and the "*generational turnover effect*" does not prevent the economy from coming back to its initial level of after-tax interest rate and of investment in education. When the education good is introduced, the negative influence of the environmental policy appears, for the reason mentioned in the previous paragraph.

Finally, we demonstrate that the BGP optimal growth rate is independent from the environmental care despite the finite lifetime. This is because the central planner internalizes the "*generational turnover effect*" . Such a result is similar to the one found by Vellinga (1999) with infinite lifetime and a single representative agent: environmental care does not influence optimal growth when utility is additive and pollution does not influence the ability of agents to be educated.

In section 2, we expose the model. Section 3 investigates the steady-state equilibrium. Section 4 examines the transitional dynamics of the model and compare the transitional effects of a tighter environmental policy on the economy when lifetime is finite and infinite, using numerical simulations. Section 5 discusses results, assuming alternative assumptions about production factors in education and the source of pollution. Section 6 examines optimal growth and the optimal environmental tax. Section 7 concludes.

2 The model

Let's consider a Yaari (1965)-Blanchard (1985) overlapping generations model with human capital accumulation and environmental concerns. Time is continuous. Each individual born at time s faces a constant probability of death per unit of time $\lambda \geq 0$. Consequently his life expectancy is $1/\lambda$. When λ increases, the life span decreases. At time s, a cohort of size λ is born. At time $t \geq s$, this cohort has a size equal to $\lambda e^{-\lambda(t-s)}$ and the constant population is equal to $\int_{-\infty}^{t} \lambda e^{-\lambda(t-s)} ds = 1$. There are insurance companies and there is no bequest motive.[5]

The expected utility function of an agent born at $s \leq t$ is:[6]

$$\int_{s}^{\infty} \left[\log c(s,t) - \zeta \log \mathcal{P}(t)\right] e^{-(\varrho+\lambda)(t-s)} dt \tag{1}$$

where $c(s,t)$ denotes consumption in period t of an agent born at time s, $\varrho \geq 0$ is the rate of time preference and $\zeta > 0$ measures the weight in utility attached to the environment, that is environmental care.

The representative agent can increase his stock of human capital by devoting time to schooling, according to Lucas (1988):[7]

$$\dot{h}(s,t) = B\left[1 - u(s,t)\right] h(s,t) \tag{2}$$

where B is the efficiency of schooling activities, $u(s,t) \in]0,1[$ is the part of human capital allocated to productive activities at time t for the generation born at s and $h(s,t)$ is the stock of human capital at time t of an individual born at time s. Note that we make no assumption about the influence of pollution on individual human capital accumulation.

[5]The first assumption is made because here death may be interpreted as the termination of a family dynasty and therefore adults who die do not care about what occurs beyond their death. The second assumption is made to avoid unintented bequests.

[6]We use logarithmic utility for the sake of simplicity. In appendix D, we demonstrate that our results remain valid when the intertemporal elasticity of substitution of the consumption is different from unity.

[7]In appendix B, we develop the model with a more general function of human capital accumulation whose results are discussed in the following sections. For the ease of the exposition and the comprehension of the economic mechanisms we choose to examine first the Lucas (1988) specification. Appendix B and B1 demonstrate in details how the main results for this specification are obtained.

Households face the following budget constraint:

$$\dot{a}(s,t) = [r(t) + \lambda]\, a(s,t) + u(s,t)h(s,t)w(t) - c(s,t) \tag{3}$$

where $a(s,t)$ is the financial wealth in period t and $\omega(t)$ represents the wage rate per effective unit of human capital $u(s,t)h(s,t)$. In addition to the budget constraint, there exists a transversality condition which must be satisfied to prevent households from accumulating debt indefinitely:

$$\lim_{v\to\infty} \left[a_{s,v} e^{-(r+\lambda)(v-t)}\right] = 0$$

The representative agent chooses the time path for $c(s,t)$ and his working time $u(s,t)$ by maximizing (1) subject to (2) and (3). It yields

$$\dot{c}(s,t) = [r - \varrho]\, c(s,t) \tag{4}$$

Integrating (3) and (4) and combining the results gives the consumption at time t of an agent born at time s:

$$c(s,t) = (\varrho + \lambda)\, [a(s,t) + \omega(s,t)] \tag{5}$$

where $\omega(s,t) \equiv \int_t^\infty [u(s,\nu)h(s,\nu)w(\nu)]\, e^{-\int_t^\nu [r(\zeta)+\lambda] d\zeta} d\nu$ is the present value of lifetime earning. It also gives the equality between the rate of return to human capital and the effective rate of interest:[8]

$$\frac{\dot{w}(t)}{w(t)} + B = r(t) + \lambda \tag{6}$$

Due to the simple demographic structure, all individual variables are additive across individuals. Consequently, the aggregate consumption equals

$$C(t) = \int_{-\infty}^{t} c(s,t)\lambda e^{-\lambda(t-s)} ds = (\varrho + \lambda)\, [K(t) + \Omega(t)] \tag{7}$$

[8]The effective interest rate is the interest rate on the debt r plus the insurance premium λ the agent has to pay when borrowing (Blanchard and Fisher, 1989).

where $\Omega(t) \equiv \int_{-\infty}^{t} \omega(s,t)\lambda e^{-\lambda(t-s)} ds$ is aggregate human wealth in the economy. The aggregate stock of physical capital is defined by

$$K(t) = \int_{-\infty}^{t} a(s,t)\lambda e^{-\lambda(t-s)} ds$$

and the aggregate human capital is

$$H(t) = \int_{-\infty}^{t} h(s,t)\lambda e^{-\lambda(t-s)} ds, \tag{8}$$

We assume that the human capital of an agent born at current date, $h(t,t)$, is inherited from the dying generation. Because the mechanism of intergenerational transmission of knowledge is complex, we make the simplifying assumption that the human capital inherited from the dying generation is a constant part of the aggregate level of human capital such that $h(t,t) = \eta H(t)$ with $\eta \in]0,1]$ (see Song, 2002).[9]

The productive sector is competitive. The representative firm produces the final good Y with the following technology:

$$Y(t) = K(t)^{\alpha} \left[\int_{-\infty}^{t} u(s,t)h(s,t)\lambda e^{-\lambda(t-s)} ds \right]^{1-\alpha}, \qquad 0 < \alpha < 1$$

with $Y(t)$ being the aggregate final output. $\int_{-\infty}^{t} u(s,t)h(s,t)\lambda e^{-\lambda(t-s)} ds$ is the amount of the aggregate stock of human capital used in production.

Following Gradus and Smulders (1993), pollution flow is assumed to increase with the stock of physical capital K and reduces with private abatement activities D (made by the firms and which consumes final output one for one):

$$\mathcal{P}(t) = \left[\frac{K(t)}{D(t)} \right]^{\gamma}, \qquad \gamma > 0 \tag{9}$$

We assume that the government implements an environmental policy to incite firms to reduce their net flow of pollution. To do so, the government taxes the net flow of pollution by the firms and transfers to them the fruit of the taxes to fund their abatement

[9] We assume that η could be equal to unity, that is the total aggregate level of human capital is inherited from the dying generation. Because population is constant and normalized to unity, this assumption could be viewed alternatively as the fact that the mean (or per capita) aggregate level of human capital is inherited from the dying generation by each newborn.

activities. Consequently, the representative firm under perfect competition pays a pollution tax on its net pollution $\mathcal{P}(t)$ and it chooses its abatement activities $D(t)$ (whose cost equals $D(t)$) and the amount of factors which maximize its profits $\pi(t) = Y(t) - r(t)K(t) - w(t)\left[\int_{-\infty}^{t} u(s,t)h(s,t)\lambda e^{-\lambda(t-s)}ds\right] - \vartheta(t)\mathcal{P}(t) - D(t) + T(t)^p$ where $\vartheta(t)$ is the pollution tax rate and $T^p(t)$ denotes transfers from the public sector with $T^p(t) = \vartheta(t)\mathcal{P}(t)$. The representative firm takes as given these transfers and pays each production factor at its marginal productivity to maximize profit:

$$r(t) = \alpha\frac{Y(t)}{K(t)} - \vartheta(t)\gamma\frac{\mathcal{P}(t)}{K(t)}$$

$$w(t) = (1-\alpha)K(t)^{\alpha}\left[\int_{-\infty}^{t} u(s,t)h(s,t)\lambda e^{-\lambda(t-s)}ds\right]^{-\alpha} \tag{10}$$

$$D(t) = \vartheta(t)\gamma\mathcal{P}(t) \tag{11}$$

From equations (9) and (11), we obtain:

$$\mathcal{P}(t) = \left[\gamma\frac{\vartheta(t)}{K(t)}\right]^{-\gamma/(1+\gamma)} \tag{12}$$

In the long-run, the net flow of pollution will be constant because K and D will evolve at the same growth rate (see section 3 below). As a result, from equation (12), the environmental tax must rise over time because the physical capital stock accumulates over time (see Hettich, 1998). Intuitively, $\vartheta(t)$ increases over time to encourage firms to increase abatement activities to limit pollution which rises with the physical capital stock. Consequently, we define $\tau \equiv \vartheta(t)/K(t)$, the environmental tax normalized by the physical capital stock and we obtain:

$$\mathcal{P} = \Phi(\tau)^{-\gamma} \tag{13}$$

$$D(t) = \Phi(\tau)K(t)$$

with $\Phi(\tau) \equiv (\gamma\tau)^{1/(1+\gamma)}$. Because τ is fixed by the government and therefore has no transitional dynamics, $\mathcal{P}$ is independent of time.

3 The general equilibrium and the balanced growth path

The final market clearing condition is:

$$Y(t) = C(t) + \dot{K}(t) + D(t),$$

Differentiating (8) with respect to time and using the fact that $u(s,t) = u(t)$,[10] the aggregate accumulation of human capital is:

$$\dot{H}(t) = B\left[1 - u(t)\right]H(t) - (1-\eta)\lambda H(t) \tag{14}$$

The first term in the right-hand side of the equation represents the increase in the aggregate human capital due to the investment of each alive generation in education at time t. The second term represents the loss of human capital due to the vanishing of dying generation net from the intergenerational transmission of human capital. Indeed, on the one hand, a part λ of the living cohort born at s with a stock of human capital equal to $h(s,t)\lambda e^{-\lambda(t-s)}$ vanishes reducing growth by $\lambda\int_{-\infty}^{t} h(s,t)\lambda e^{-\lambda(t-s)}ds = \lambda H(t)$ when all generations are aggregated. On the other hand, at the same time, a new cohort of size λ appears, adding $\lambda h(t,t)$ to growth, with $h(t,t) = \eta H(t)$ and $\eta \in]0,1]$ (see above). This net loss reduces the aggregate accumulation of human capital.

Differentiating (7) with respect to time gives

$$\frac{\dot{C}(t)}{C(t)} = \frac{\dot{c}(s,t)}{c(s,t)} - \frac{1}{C(t)}\left[\lambda C(t) - \lambda c(t,t)\right] \tag{15}$$

Aggregate consumption growth differs from individual consumption growth by the term into brackets $-\left[\lambda C(t) - \lambda c(t,t)\right]$ which represents what Heijdra and Ligthart (2000) called the "*generational turnover effect*". This effect appears because at each date a cross-section of the existing population dies (reducing aggregate consumption growth by $\lambda C(t)$) and a new generation is born (adding $\lambda c(t,t)$). Because new agents born without financial assets, their consumption

[10] Using (10), the equalization of the rates of return given by equation (6) implies that the rate of return to human capital is independent of s, therefore all individuals allocate the same effort to schooling: $u(s,t) = u(t)$.

$c(t,t)$ is lower than the average consumption $C(t)$ and therefore the "*generational turnover effect*" reduces the growth rate of the aggregate consumption.

Using the expression of $dK(t)/dt$, $d\Omega(t)/dt$ and equation (4) we obtain (see appendix A):

$$\dot{C}(t)/C(t) = r(t) - \varrho - (1-\eta)\lambda - \eta\lambda(\varrho+\lambda)K(t)/C(t) \tag{16}$$

The *generational effect* rises with the probability to die λ: on one hand, agents die at a higher frequency (that increases the generational turnover) and on the other hand the propensity to consume out of wealth $\varrho + \lambda$ increases due to the shorter horizon. Compared with the case where there is no human capital accumulation, a new term η appears that captures the fact that newborns inherit from the dying generation only a part $\eta \in]0,1]$ of the aggregate human-wealth (see appendix A) and not the total amount (like in Yaari (1965)-Blanchard (1985) model).

Using previous results,[11] we can write the dynamics of the model as:

$$\begin{aligned}
\dot{x}(t) = &\ \left\{[\alpha-1]\,(b(t)u(t))^{1-\alpha} - \varrho - (1-\eta)\lambda - \eta\lambda(\varrho+\lambda)x(t)^{-1} + x(t)\right\}x(t) \\
\dot{b}(t) = &\ \left\{B\,[1-u(t)] - (1-\eta)\lambda - (b(t)u(t))^{1-\alpha} + x(t) + \Phi(\tau)\right\}b(t) \\
\dot{u}(t) = &\ \left\{(\alpha^{-1}-1)\,B + (\alpha^{-1}-1)\Phi(\tau) + Bu(t) - (\alpha^{-1}+\eta-1)\lambda - x(t)\right\}u(t)
\end{aligned} \tag{17}$$

where $x(t) \equiv C(t)/K(t)$ and $b(t) \equiv H(t)/K(t)$.

Along the balanced growth path, C, K, H, D and Y evolve at a common positive rate of growth (denoted $g^\star$, where a $\star$ means "*along the BGP*") and the allocation of human capital across sectors is constant. As a consequence, along the balanced growth path $\dot{x} = \dot{b} = \dot{u} = 0$, $x = x^\star$, $b = b^\star$, $u = u^\star$ and $g^\star > 0$.

Proposition 1. *Under the conditions that along the Balanced growth path, the rate of growth must be positive and can not exceed the maximum feasible rate, there exists a unique* $u^\star \in]\underline{u}, \bar{u}[$ *(with* $\underline{u} \equiv \frac{\varrho+\lambda}{B}$, $\bar{u} \equiv 1 - \frac{(1-\eta)\lambda}{B}$, *and* $0 < \underline{u} < \bar{u} < 1$*) solving* $\Gamma(u;\tau) = 0$ *where* $\Gamma(u;\tau)$ *is defined*

[11] See details of the calculation in appendix B and B1.

as follows

$$\Gamma(u;\tau) \equiv [Bu - \lambda - \varrho] \times \left\{ \left[\alpha^{-1} - 1 + u\right] B + (\alpha^{-1} - 1)\Phi(\tau) - \left(\alpha^{-1} - 1 + \eta\right) \lambda \right\} - \eta\lambda(\varrho + \lambda).$$

Furthermore, $u^\star$ is a decreasing function of τ denoted by $\mathcal{U}(\tau)$ with $\mathcal{U}'(\tau) < 0$, and the BGP equilibrium is saddle-path stable.

Proof. See Appendix B, B1 and C. ■

The conditions that along the BGP the rate of growth must be positive (here $u^\star < \bar{u}$) and can not exceed the maximum feasible rate of growth (here $u^\star > \underline{u}$) are conventional in the Lucas (1988) human capital accumulation model.[12] The negative influence of the environmental taxation on $u^\star$ is explained below (after Proposition 3).

From (17), using the fact that along the BGP $\dot{b} + \dot{u} = 0$, we can express the human capital to physical capital ratio H/K along the balanced growth path as:

$$b^\star = \left[\alpha^{-1} \left(B - \lambda + \Phi(\tau)\right)\right]^{1/(1-\alpha)} \mathcal{U}(\tau)^{-1} > 0$$

The increase in the environmental tax leads the final production to be less intensive in physical capital and more intensive in human capital because it increases the cost of physical capital: $b^\star$ rises. The aggregate consumption to physical capital ratio C/K along the balanced growth path is given by (see appendix B1):

$$x^\star = \frac{\lambda\eta(\varrho + \lambda)}{B\mathcal{U}(\tau) - \lambda - \varrho} > 0$$

and is an increasing function of the environmental tax rate.

Finally, the growth of the market economy along the BGP is:

$$g^\star = B\left[1 - \mathcal{U}(\tau)\right] - (1 - \eta)\,\lambda. \tag{18}$$

[12]See appendix B for details and the textbook by Barro and Sala-I-Martin (1995) for example.

Proposition 2. *When lifetime is finite, the environmental policy has a positive impact on the Balanced growth path rate of growth driven by human capital accumulation à la Lucas (1988), while pollution does not influence educational activities, labor supply is inelastic and human capital does not enter the utility function.*

Proof. It comes directly from proposition 1 and equation (18). ■

Consequently, by assuming finite lifetime, it is possible to implement a win-win environmental policy in a Lucas (1988) growth model. Furthermore, when the horizon extends (λ decreases), the allocation of human capital into final production $u^\star$ drops: agents invest more in human capital, as demonstrated in Appendix B1. And when lifetime is infinite, $\lambda = 0$, the allocation of human capital into the production $u^\star$ is independent from τ along the balanced growth path, but $b^\star$ and $x^\star$ are increasing functions of τ (see appendix B2.).

Proposition 3. *The ageing of the population (a lower λ) reduces the influence of the environmental policy on growth in the Lucas (1988) model with finite lifetime. When lifetime is infinite, we find the conventional result of the Lucas (1988) model: the BGP rate of growth is not affected by the environmental policy.*

Proof. See Appendix B1 and B2. ■

To understand the mechanisms that explain Propositions 1, 2 and 3, let us consider first the case where lifetime is infinite ($\lambda = 0$) and there is a single representative household and the "*generational turnover effect*" is absent. In such a case, the environmental policy, through a tighter environmental tax, has two effects. First, a crowding-out effect, due to the rise of abatement expenditures, reduces consumption and investment. Second, a factorial reallocation effect leads to a production more intensive in human capital: there is a substitution of the pollutant factor (physical capital) by the "clean" factor (human capital). Indeed, the rise of the environmental tax lowers instantaneously the after-tax interest rate. The rewards to physical capital becomes lower than the rewards to human capital and agents instantaneously reallocate their human capital in educational activities: u falls to equalize the returns between physical capital accumulation and human capital accumulation. Human capital accumulation is boosted while the growth rate of physical capital drops and output production becomes more intensive in the non-pollutant factor of production: the human capital. Therefore, the human capital to physical capital ratio b increases. As a result, the after-tax interest rate begins to increase and gradually agents reallocate their ressources to physical capital (u gradually increases while the substitution between physical capital and human capital continues because the after-tax interest rate remains lower than the returns to human capital along the BGP equal to B). The increase in b and u stops when the after-tax interest rate is back to its initial value B. That occurs for the initial value of $u^\star$ and for a higher value of b (with respect to its initial value $b^\star$). Because the aggregate consumption rate of growth only depends on the after-tax interest rate, its after increasing tax value equals its inital value, and the growth rate of human capital and physical capital remain unchanged. Note that u along the new BGP remains unchanged with respect to its initial situation because otherwise the growth rate of human capital would be different from the growth rate of consumption and physical capital and as a result b would not be constant.[13]

When agents have finite lives ($\lambda > 0$), the tighter environmental policy has a third impact: a "*generational turnover effect*" that affects the aggregate consumption rate of growth. Indeed, with finite lifetime, the aggregate consumption rate of growth differs from the individual consumption rate of growth $r - \varrho$, by the "*proportionnal*" difference between average consumption and consumption by newly born households $\left[C(t) - c(t,t)\right]/C(t)$ (see equation 15). And this difference, that reduces the aggregate consumption rate of growth, depends positively on the physical capital to aggregate consumption ratio x^{-1} (that is for a given after-tax interest rate, the aggregate consumption growth rises with x, see equation 16). It explains the positive impact of the environnmental taxation of the BGP growth with finite lifetime. When b and u

[13] If u is higher than $u^\star$, human capital accumulation is lower than physical capital accumulation and therefore, b decreases. As a result, the interest rate decreases and agents rise their investment in human capital: u reduces.

increases after the shock and the after-tax interest rate back to its initial level, the individual consumption rate of growth $r - \varrho$ also back to its initial level, but it is not the case for the aggregate consumption rate of growth. Indeed, the substitution of the physical capital by the human capital increases the aggregate consumption to physical capital ratio x leading to an aggregate consumption rate of growth (when lifetime is finite) higher than its initial value. In order to restore the equalization of the growth rates along the BGP, the physical capital rate of growth increases. *Ceteris Paribus*, the human capital to physical capital ratio b diminishes and as a result the after-tax interest rate falls: agents allocate a part of their resources from final output to human capital accumulation (u diminishes) and the interest rate back to its inital value. As a result, the aggregate consumption growth, the physical capital and the human capital growth are equalized and higher than their initial value along the new BGP equilibrium. The allocation of human capital into the manufacturing sector u along the new BGP equilibrium is lower than its initial value: the environmental taxation reduced it.

The previous explanation enables us to understand Proposition 3. The longer the horizon (the lower λ), the lower the negative influence of x on the aggregate consumption rate of growth (see equation 16), that is the lower the difference between the aggregate consumption rate of growth and the individual consumption rate of growth. Therefore, the lower the increase in the physical capital and human capital accumulation to equalize the rates of growth along the BGP. As a result, the lower the fall in u to obtain such an increase. For a given rise of the environmental tax, the allocation of human capital to the output sector along the BGP ($u^{\star}$) reduces less for a lower λ than for a higher λ.

4 The transitional dynamics

In this section we investigate the trajectory of the economy out of the steady-state using the time-elimination method. We compare the influence of the environmental policy during the transition both when lifetime is finite and when lifetime is infinite.

Due to the complexity of the model, we use numerical simulations to look at the transitional dynamics, and especially the transitional impact of the environmental tax. We calibrate the model to obtain realistic values of the growth rate of GDP and the probability of death for the US economy. From the *World Development Indicators 2005* by the World Bank, life expectancy was 77.4 years in 2003, the growth rate was 3.3% during the period 1990-2002 and a public health expenditures as percentage of GDP was 6.55%. Since the expected lifetime is the reverse of the probability of death per unit of time λ, we want λ to be close to $1/77.4 = 0.0128$. We adjust other variables to obtain such values for our benchmark case.

Table 1 summarizes the benchmark value of parameters and Table 2 summarizes the exercise of comparative statics for log utility.

α	η	ϱ	B	γ	$\lambda^{(1)}$	$\lambda^{(2)}$
0.3	0.85	0.025	0.075	0.3	0.0128	0

$^{(1)}$ finite lifetime $^{(2)}$ infinite lifetime

Table 1. Benchmark value of parameters

	Finite lifetime				Infinite lifetime	
	$\lambda = 0.0128$		$\lambda = 0.0200$			
	$\tau = 0.01$	$\tau = 0.1$	$\tau = 0.01$	$\tau = 0.1$	$\tau = 0.01$	$\tau = 0.1$
$g^\star$	3.32%	3.40%	2.29%	2.45%	5%	5%
$\mathcal{P}^\star$	3.82	2.25	3.82	2.25	3.82	2.25
$u^\star$	0.5313	0.5206	0.6545	0.6323	1/3	1/3
$r^{\star(1)} + \lambda$	0.075	0.075	0.075	0.075	0.075	0.075
$x^{\star(2)}$	0.2009	0.3305	0.1872	0.3160	0.2268	0.3572
$b^{\star(3)}$	0.2532	0.5790	0.1774	0.4394	0.5073	1.0345

$^{(1)}\alpha Y/K - \Phi(\tau)$ $^{(2)}C/K$ $^{(3)}H/K$

Table 2. The increase in the environmental tax along the BGP

Graph 1 and Graph 2 draw the temporal evolution of the main variables towards the new steady-state when an unanticipated increase in the environmental tax is implemented by the government, respectively for finite and infinite lifetime.

What differs between finite and infinite lifetime is mainly the size of the variations and the fact that all variables tend towards a new steady-state value in the finite lifetime case. Let consider this case to look at the out-of steady-state evolution of variables (Graph 2).

First of all, an increase in the environmental taxation lowers instantaneously the after-tax interest rate (Figure 4 in Graph 2) and drops the growth rate of physical capital which becomes negative (Figure 8 in Graph 2). Because the returns to education becomes higher, agents allocate their resource towards human capital accumulation: the allocation of human capital into the manufacturing sector falls (Figure 1 in Graph 2) while the growth rate of human capital jumps (Figure 7 in Graph 2). The manufacturing sector becomes more intensive in human capital and the ratio human capital to physical capital (b) rises (Figure 2 in Graph 2). The fall in u and the increase in b contributes to rise the interest rate and reduces the incentives of agents to allocate human capital into the educational sector: u rises while the increase in the human capital physical capital ratio decelerates. This continues up to the after-tax interest rate comes to its initial value (equal to B). While the allocation of human capital into production and the growth rates of the physical capital, the human capital, consumption and final output back to their initial steady-state value when the lifetime of agents is infinite (see Graph 1), this is not the case when the lifetime is finite because the difference between aggregate consumption and consumption by newly born household is reduced because it depends on x^{-1} (Figure 6 *called "Diff Intergen"* Graph 2). As explained at the end of section 3, that leads agents to allocate more human capital into the educational sector ($u\star$ is lower) and that promotes human capital accumulation, aggregate consumption growth and aggregate output growth along the new BGP equilibrium (see Figures 7,8,9,10 Graph 2).[14]

5 Discussion

In the previous sections, we demonstrated that the "*generational turnover effect*" , due to the appearance of newborn at each date and the death of a part of the population, leads to a positive impact of the environmental policy in the Lucas (1988) model without assuming that

[14]Remark that the externality in the aggregate human capital accumulation due to the generational turnover $(1-\eta)\lambda H(t)$ is not required to obtain our result. Letting $\eta = 1$ in the expression of $\Gamma(u;\tau)$ (in Proposition 1), the allocation of human capital into production $u^{\star}$ remains negatively influenced by the environmental tax. The only effect is a higher growth rate along the BGP.

a better environmental quality enhances ability to learn, that labor supply is elastic or that education enters the utility function as a consumption good.

Nevertheless, according to empirical evidence, education is not only the fruit of a private time investment. It requires either physical capital (see for example King and Rebelo, 1990) or public expenditures (see for example Glomm and Ravikumar, 2001). Furthermore, as highlighted by Hettich (1998), the assumption about the source of pollution may completely change the impact of the environmental policy on growth. Do our results continue to hold when the technology in the educational sector is modified in such a way and/or when an alternative source of pollution is introduced?

5.1 Education good in human capital accumulation

Following Rebelo (1991), we consider that, besides time, education requires an educational input produced with physical and human capital. To simplify things we suppose that it is produced in the same way than the final output. The fact that households buy an educational input will modify their decisions to invest their time to educational activities and the influence of the environmental policy on human capital accumulation and growth.

At time t, each agent born at $s \leq t$ buy $z(s,t)$ units of final output which increase the productivity of the time that they invest in education, such as:

$$\dot{h}(s,t) = B\left[(1-u(s,t))h(s,t)\right]^{1-\delta} z(s,t)^{\delta} \tag{19}$$

with $\delta \in [0,1[$,[15] and

$$\dot{a}(s,t) = \left[r(t)+\lambda\right]a(s,t) + u(s,t)h(s,t)w(t) - c(s,t) - z(s,t)$$

Utility maximization implies that $u(s,t)$ and the ratio $\dfrac{z(s,t)}{h(s,t)}$ are independent from s (conveniently, we denote $\tilde{z}(t) \equiv \dfrac{z(s,t)}{(1-u(s,t))h(s,t)}$). Furthermore, it gives (see appendix B)

$$\tilde{z}(t) = \frac{\delta}{1-\delta}w(t) \tag{20}$$

[15] When $\delta = 0$, we obtain the Lucas (1988) human capital accumulation of the previous sections. The demonstration is detailed in appendix B.

and the equalization of the returns to education

$$(1-\delta)\frac{\dot{w(t)}}{w(t)} + B(1-\delta)^{1-\delta}\delta^{\delta}w(t)^{\delta} = r(t) + \lambda$$

The aggregate accumulation of human capital is then written as

$$\dot{H}(t) = B(1-u(t))\tilde{z}(t)^{\delta} - (1-\eta)\lambda$$

The amount of final output used as educational good is $Z(t) \equiv \int_{-\infty}^{t} z(s,t)\lambda e^{-\lambda(t-s)}ds = \tilde{z}(t)(1-u(t))H(t)$ from previous results. Consequently, from (20) it can be expressed as a function of final output:

$$Z(t) = \Delta\left(\frac{1-u(t)}{u(t)}\right)Y(t)$$

with $\Delta \equiv \frac{\delta(1-\alpha)}{1-\delta}$, and the market clearing condition becomes:

$$\left(1+\Delta\left(1-\frac{1}{u(t)}\right)\right)Y(t) = \dot{K}(t) + C(t) + \Phi(\tau)K(t)$$

Consequently, the dynamical system is summarized by (see Appendix B)

$$\begin{aligned}
\dot{x}(t) = &\ \left\{\left[\alpha - \left(1+\Delta\left(1-\frac{1}{u(t)}\right)\right)\right](b(t)u(t))^{1-\alpha} - \varrho - (1-\eta)\lambda - \eta\lambda(\varrho+\lambda)x(t)^{-1} + x(t)\right\}x(t) \\
\dot{b}(t) = &\ \{(1-u(t))B\Delta^{\delta}(b(t)u(t))^{-\alpha\delta} \\
&-(1-\eta)\lambda + x(t) + \Phi(\tau) - \left(1+\Delta\left(1-\frac{1}{u(t)}\right)\right)(b(t)u(t))^{1-\alpha}\Big\}\, b(t) \\
\dot{u}(t) = &\ \{[\alpha^{-1} - 1 + u(t)]\,B\Delta^{\delta}(b(t)u(t))^{-\alpha\delta} + ((\alpha(1-\delta))^{-1} - 1)\,\Phi(\tau) \\
&+\Delta\left[\left(1-\frac{1}{u(t)}\right) - \frac{1}{1-\alpha}\right](b(t)u(t))^{1-\alpha} - ((\alpha(1-\delta))^{-1} + \eta - 1)\,\lambda - x(t)\Big\}\, u(t)
\end{aligned}$$

Along the steady-state, from the last equation of the dynamical system, we obtain

$$(1-\delta)B\Delta^{\delta}(b^{\star}u^{\star})^{-\alpha\delta} = \alpha(b^{\star}u^{\star})^{1-\alpha} - \Phi(\tau) + \lambda \tag{21}$$

where the left-hand side is the returns to human capital accumulation along the BGP and the right-hand side is the effective interest rate (the returns to physical capital accumulation), also evaluated along the BGP. This relation states $b^\star u^\star$ as an increasing function of τ ($d\Phi(\tau)/d\tau > 0$), denoted by $\mathcal{R}(B,\tau)$ with $d\mathcal{R}(B,\tau)/d\tau > 0$ and $d\mathcal{R}(B,\tau)/dB > 0$. The two remaining equations of the dynamical system evaluated at the steady-state ($\dot{x}(t) = 0$ and $\dot{b}(t) = 0$) enable us to write the following proposition.

Proposition 4. [Generalization of Proposition 1 for $\delta \in [0,1[$.] *Under the conditions that along the Balanced growth path, the rate of growth must be positive and can not exceed the maximum feasible rate, there exists a unique $u^\star \in]\underline{u}_\delta, \bar{u}_\delta[$ with ($\underline{u}_\delta \equiv \delta + \frac{\varrho+\lambda}{B\Delta^\delta \mathcal{R}(B,\tau)^{-\alpha\delta}}$, $\bar{u}_\delta \equiv 1 - \frac{(1-\eta)\lambda}{B\Delta^\delta \mathcal{R}(B,\tau)^{-\alpha\delta}}$, and $0 < \underline{u}_\delta < \bar{u}_\delta < 1$), solving $\Gamma_\delta(u;\tau) = 0$ where $\Gamma_\delta(u;\tau)$ is defined as follows*

$$\begin{aligned}\Gamma_\delta(u;\tau) \equiv & \left[(u-\delta)B\Delta^\delta \mathcal{R}(B,\tau)^{-\alpha\delta} - \lambda - \varrho\right] \times \\ & \left\{(u-\delta)B\Delta^\delta \mathcal{R}(B,\tau)^{-\alpha\delta} - \eta\lambda + \frac{(1-\alpha)(u^\star-\delta)}{(1-\delta)u}\mathcal{R}(B,\tau)^{1-\alpha}\right\} - \eta\lambda(\varrho+\lambda)\end{aligned}$$

with $\mathcal{R}(B,\tau) = b^\star u^\star$ solution of equation (21)*. The BGP equilibrium is saddle-path stable.*

Proof. See Appendix B and C. ∎

The aggregate growth rate in the economy is given by:

$$g^\star = B(1-u^\star)\Delta^\delta \mathcal{R}(B,\tau)^{-\alpha\delta} - (1-\eta)\lambda$$

Conversely to the case where only time is used as input in education, here the influence of the environmental taxation on the allocation of time in education $u^\star$ is not clear-cut. When lifetime is infinite ($\lambda = 0$), we demonstrate that $u^\star$ rises with τ (see Appendix B2), because the tighter environmental policy not only crowds-out consumption and physical capital accumulation, but also the part of output allocated to the education sector Z. As a result, the rewards to education falls below its initial value and the agents reallocate their time to production to compensate the decrease in their consumption. When lifetime is finite ($\lambda > 0$), the aforementioned effect exists besides the "*generational turnover effect*" which operates in the opposite way. Is the "*generational turnover effect*" high enough to compensate or to offset the crowding-out effect?

That is the question we investigate in the following, using numerical simulations because the global impact of the environmental tax is very cumbersome to derive analytically.

Especially, we look at the impact of an increase in environmental taxation for different values of δ, the part of the education good in human capital accumulation, insofar as we demonstrated that the environmental policy enhances growth when only time is used as input for education (that is $\delta = 0$) and because we expect that environmental policy is harmful for growth when only final output is used to increase human capital (that is $\delta = 1$).[16] We report results in table 3.[17]

Table 3. Impact of the environmental policy according to δ.

	$\delta = 0.01$		$\delta = 0.06$		$\delta = 0.1$		$\delta = 0.5$	
	$\tau = 0.01$	$\tau = 0.1$	$\tau = 0.01$	$\tau = 0.1$	$\tau = 0.01$	$\tau = 0.1$	$\tau = 0.01$	$\tau = 0.065$
$g^\star$	2.92%	2.99%	1.8766%	1.8821%	1.3825%	1.3378%	0.4071%	0.0205%
$\mathcal{P}^\star$	3.82	2.25	3.82	2.25	3.82	2.25	3.82	2.48
$u^\star$	0.5663	0.5556	0.6812	0.6750	0.7480	0.7478	0.9362	0.9747
$r^\star + \lambda$	0.0711	0.0709	0.0610	0.0600	0.0562	0.0546	0.0470	0.0419
$x^\star$ (C/K)	0.19061	0.3187	0.1645	0.2874	0.1530	0.2736	0.1295	0.2051
$b^\star$ (H/K)	0.2198	0.5183	0.1461	0.3745	0.1181	0.3156	0.0725	0.1484

Positive effect on growth *Negative effect on growth*

The first insight of our simulations is that the positive effect of the environmental policy due to finite lifetime exists, for the parameters values chosen. The second insight is that this positive effect offsets the crowding-out effect only if the part of the education good in human capital accumulation (δ) is small enough.

Consequently, it comes the following proposition.

Proposition 5. *Proposition 2 still holds when an education good (produced with human and physical capital) is introduced in the technology of education, only if the part of this education good in human capital accumulation is small enough.*

[16] In such a case, the technology to accumulate human capital is similar to the technology of final output production and physical capital accumulation except the parameter B (see equation 19).

[17] Note that in our numerical simulation, when $\delta = 0.5$, $u^\star \notin]\underline{u}_\delta, \bar{u}_\delta[$ and $g^\star < 0$ when $\tau > 0.065$.

Proof. See numerical simulations above. ■

The economic mechanisms behind this result are similar to those previously stated (See Graph 3 and Graph 4 for the transitional dynamics). In the presence of an education good produced with final output, the tighter environmental tax leads to a further crowding-out effect that reduces the amount of education good in human capital accumulation. When the relative part of this education good in the human capital accumulation (with respect to the time of education) becomes important, this crowding-out effect offsets the positive effect arising from the finite lifetime and the "*generational turnover effect*" . The BGP rate of growth drops.

5.2 Alternative specification of pollution

As demonstrated by Hettich (1998), the specification of pollution may modify the impact of the environmental taxation on growth. In a Uzawa-Lucas (1988) model with elastic labor supply, he finds that the environmental taxation enhances growth when the source of pollution is the stock of physical capital. Nevertheless, assuming that the source of pollution is rather final output, he obtains no effect of the environmental taxation. Indeed, when final production is the source of pollution, the environmental tax not only impacts the interest rate, but also the wage rate, erasing the positive effect which transits through the elastic labor supply. In this section, we re-examine our previous results assuming that the source of pollution is final output rather than physical capital.

In this case, we have

$$\mathcal{P}(t) = \left[\frac{Y(t)}{D(t)}\right]^{\gamma}$$

Therefore, profit maximization in the final output production leads to

$$r(t) = \left(1 - \vartheta(t)\gamma\frac{\mathcal{P}(t)}{Y(t)}\right)\alpha\frac{Y(t)}{K(t)}$$

$$w(t) = \left(1 - \vartheta(t)\gamma\frac{\mathcal{P}(t)}{Y(t)}\right)(1-\alpha)K(t)^{\alpha}\left[\int_{-\infty}^{t} u(s,t)h(s,t)\lambda e^{-\lambda(t-s)}ds\right]^{-\alpha}$$

and

$$D(t) = \vartheta\gamma\mathcal{P}(t)$$

Defining $\hat{\tau} \equiv \vartheta(t)/Y(t)$, the environmental tax normalized by the final output production we obtain:

$$\mathcal{P} = \Phi(\hat{\tau})^{-\gamma}$$

$$D(t) = \Phi(\hat{\tau})Y(t)$$

with $\Phi(\hat{\tau}) \equiv (\gamma\hat{\tau})^{1/(1+\gamma)}$.

The final market clearing condition becomes:

$$[1 - \Phi(\hat{\tau})]\, Y(t) = C(t) + \dot{K}(t)$$

and the dynamical system is, for $\delta \in [0, 1[$:

$$\begin{aligned}\dot{x}(t) = \Big\{ \Big[(1 - \Phi(\hat{\tau}))\,(\alpha - 1) - \Delta\left(1 - \frac{1}{u(t)}\right) \Big]\, (b(t)u(t))^{1-\alpha} \\ -\varrho - (1-\eta)\lambda - \eta\lambda(\varrho + \lambda)x(t)^{-1} + x(t) \big\}\, x(t)\end{aligned}$$

$$\begin{aligned}\dot{b}(t) = \big\{ B(1 - u(t))[\Delta(b(t)u(t))^{-\alpha}]^{\delta} - (1-\eta)\lambda \\ - \left(1 - \Phi(\hat{\tau}) + \Delta\left(1 - \frac{1}{u(t)}\right)\right) (b(t)u(t))^{1-\alpha} + x(t) \Big\}\, b(t)\end{aligned}$$

$$\dot{u}(t) = \Big\{ \alpha^{-1}\left[(1-\delta)B\Delta^{\delta}(b(t)u(t))^{-\alpha\delta} - \alpha(1 - \Phi(\hat{\tau}))(b(t)u(t))^{1-\alpha} - \lambda\right] - \dot{b}/b \Big\}\, u(t)$$

Along the BGP we have $\dot{u} = 0$ and we obtain the equalization of returns

$$(1-\delta)B\Delta^{\delta}(b^{\star}u^{\star})^{-\alpha\delta} = (1 - \Phi(\hat{\tau}))\alpha(b^{\star}u^{\star})^{1-\alpha} + \lambda \tag{22}$$

This relation states $b^{\star}u^{\star}$ as an increasing function of $\hat{\tau}$ denoted by $\hat{\mathcal{R}}(B, \hat{\tau})$ with $d\hat{\mathcal{R}}(B, \hat{\tau})/d\hat{\tau} > 0$ and $d\hat{\mathcal{R}}(B, \hat{\tau})/dB > 0$, for $\delta \in [0, 1[$. Finally using $\dot{x} = 0$ and $\dot{b} = 0$, we find that there exists a unique $u^\star \in]\underline{u}_\delta, \bar{u}_\delta[$ solution of $\Upsilon_\delta(u;\hat{\tau}) = 0$ where $\Upsilon_\delta(u;\hat{\tau})$ is defined as (see Appendix E):

$$\Upsilon_\delta(u;\hat{\tau}) \equiv \left[(u-\delta)B\Delta^\delta\hat{\mathcal{R}}(B,\hat{\tau})^{-\alpha\delta} - \varrho - \lambda\right] \times$$
$$\Big\{\left[u-\delta+(\alpha^{-1}-1)(1-\delta)\right] B\Delta^\delta\hat{\mathcal{R}}(B,\hat{\tau})^{-\alpha\delta} - (\alpha^{-1}-1+\eta)\lambda$$
$$+\frac{(1-u)(1-\alpha)\delta}{(1-\delta)u}\hat{\mathcal{R}}(B,\hat{\tau})^{1-\alpha}\Big\} - \eta\lambda(\varrho+\lambda) \quad (23)$$

When $\delta = 0$, the term $\Delta^\delta\hat{\mathcal{R}}(B,\hat{\tau})^{-\alpha\delta}$ reduces to unity and the last term into braces vanishes. The expression of $\Upsilon_\delta(u;\hat{\tau})$ becomes independent from δ and $\hat{\tau}$. It comes the following proposition.

Proposition 6. *When final output is the source of pollution, the allocation of human capital to education is independent from the environmental tax level and Proposition 2 does no longer hold: the environmental policy does not affect growth in the long-run, even with finite lifetime.*

Proof. See Appendix E. ■

This result comes from the fact that the tighter environmental policy affects both the rewards to physical capital accumulation and the wage rate (that influences the rewards to human capital accumulation). When $\hat{\tau}$ rises, agents allocate instantaneously more resources to human capital accumulation and the human capital to physical capital ratio (b) increases (see Figure 2 Graph 5). Nevertheless, the decrease in the wage rate (see Figure 5 Graph 5) limits the gap between the returns to human capital and the returns to physical capital: the substitution of the physical capital by the human capital is less important with respect to the case where only physical capital is a polluting factor. As a result, the physical capital stock drops less than the aggregate consumption (that decreases due to the crowding-out effect): the aggregate consumption to physical capital output ratio (x) reduces conversely to the case where only physical capital is the source of pollution (see Figure 3 Graph 5). As a result the intergenerational effect (that depends on x^{-1}) rises (see Figure 6 Graph 5) and it reinforces the drop of the aggregate consumption growth. Finally, the wage rate stabilizes to a lower value, the increase in b rises the interest rate and leads agents to reallocate a part of their human capital to production, u

increases and backs to its initial value, like the effective interest rate $r + \lambda$, the aggregate consumption to physical capital ratio x, the aggregate consumption growth and the "*generational turnover effect*" as well (see Graph 5). The rise of the environmental tax has no permanent impact on growth because both production factors are affected by the environmental taxation and the substitution between physical capital and human capital is limited.

Nevertheless, when we consider the case where the output is used in the educational sector ($\delta > 0$), the environmental tax influences the decision to be educated and therefore the BGP rate of growth (see equation 23). Because it is difficult to find analytically whether the environmental tax $\hat{\tau}$ influences the decision to be educated $u^\star$ positively or negatively (see Appendix E), we make a numerical simulation using the calibration of section 4. We report results in table 4.

Table 4. Impact of the environmental policy according to δ.

	$\delta = 0.01$		$\delta = 0.06$		$\delta = 0.1$		$\delta = 0.5$	
	$\hat{\tau} = 0.01$	$\hat{\tau} = 0.1$	$\hat{\tau} = 0.01$	$\hat{\tau} = 0.1$	$\hat{\tau} = 0.01$	$\hat{\tau} = 0.1$	$\hat{\tau} = 0.01$	$\hat{\tau} = 0.1$
$g^\star$	2.8968%	2.8951%	1.8762%	1.8673%	1.4001%	1.3872%	0.6116%	0.5610%
$\mathcal{P}^\star$	3.82	2.25	3.82	2.25	3.82	2.25	3.82	2.25
$u^\star$	0.5710	0.5704	0.6828	0.6837	0.7474	0.7489	0.9183	0.9228
$r^\star + \lambda$	0.07116	0.07114	0.06129	0.06121	0.05674	0.05661	0.04919	0.04871
$x^\star$ (C/K)	0.16660	0.16662	0.14628	0.14625	0.1364	0.1362	0.1228	0.1216
$b^\star$ (H/K)	0.1712	0.1887	0.1102	0.1193	0.0874	0.0945	0.0543	0.0577

From this Table, the following proposition holds:

Proposition 7. *When final output is the source of pollution and is used as education good in human capital accumulation, for the parameter values chosen, numerical simulations indicate that the environmental taxation negatively affects long-run human capital accumulation when lifetime is finite.*

Proof. See Table 4. ■

The explanation of this result is similar to the explanation of the negative impact of a tighter environmental policy when physical capital is the only source of pollution and the part of final output in education is high. The output used in human capital accumulation is crowded-out= by the tighter environmental tax. That leads to a global detrimental effect of the environmental policy on the BGP rate of growth.

6 Optimal growth

In the market economy analyzed in previous sections, there are two types of externalities. The first one comes from environmental preference and the source of pollution. The second one is related to finite lifetime that disconnects aggregate consumption and aggregate human capital growth to individual consumption and individual human capital growth. With no public intervention, final producers do not internalize the negative impact of their pollution flow on utility and they would pollute so much that the environmental quality would decline to unsustainable low levels. Environmental policy is needed to prevent such an occurence. Furthermore, in his decision to educate and to consume, an individual does not take into account the effects of the intergenerational transmission of knowledge in his return to education (and on the aggregate consumption rate of growth) and therefore he invests in education and saves insufficiently. Consequently, the government must also implement a policy to take into account the "*generational turnover effect*" .

In this section we examine the centralized economy when the source of pollution is the physical capital stock and education only requires time ($\delta = 0$), restricting our attention *(i)* to the influence of environmental care on optimal growth (in the vein of Vellinga, 1999) and *(ii)* to the expression of the optimal environmental tax implemented by the government to put the net flow of pollution to its optimal level along the BGP.[18]

The objective of the social planner consists in maximizing the social welfare function taking into account the intertemporal evolution of the aggregate physical and human capitals. We can

[18] For the sake of simplicity and clearness, here we do not study the implementation of the optimum, that is we do not integrate the additional tools (besides the optimal environmental policy) that lead the BGP at equilibrium to be optimal. Moreover, we do not investigate the welfare gains or costs supported by the agents during the transition, when the BGP optimal environmental tax is implemented. That would required a separate paper.

write the program as follows (see appendix F for the complete demonstration):

$$
\begin{array}{rl}
\max\limits_{\substack{c(s,t),u(s,t),D(t)\\ K(t),H(t),h(s,t)}} & \displaystyle\int_0^\infty \left\{\int_{-\infty}^t U[c(s,t),\mathcal{P}(t)]\lambda e^{-\lambda(t-s)}ds\right\} e^{-\varrho t}dt \\
\text{s.t.} & \dot{K}(t) = K(t)^\alpha[\int_{-\infty}^t u(s,t)h(s,t)\lambda e^{-\lambda(t-s)}ds]^{1-\alpha} - \int_{-\infty}^t c(s,t)\lambda e^{-\lambda(t-s)}ds - D(t) \\
& \dot{H}(t) = \int_{-\infty}^t \{B[1-u(s,t)] - (1-\eta)\lambda\}\, h(s,t)\lambda e^{-\lambda(t-s)}ds \\
& H(t) = \int_{-\infty}^t h(s,t)\lambda e^{-\lambda(t-s)}ds \\
& \mathcal{P}(t) = (K(t)/D(t))^\gamma \\
& K(t) > 0, H(t) > 0, K_0 \text{ and } H_0 \text{ given,}
\end{array}
\tag{24}
$$

with $U\left(c(s,t),\mathcal{P}(t)\right)$ defined by equation (1). The resolution of this program gives the value of the allocation of human capital to production in the long-run:

$$u_c^\star = \frac{\varrho}{B} \in]0,1[$$

and the expression of the growth rate along the BGP is:

$$g_c^\star = B - \varrho - \lambda(1-\eta) \tag{25}$$

We find the results of the Lucas (1988) model with no externality and the following proposition holds:

Proposition 8. *We obtain with finite lifetime the same result found by Vellinga (1999) with infinite lifetime and a single representative agent: when preferences are additive and the ability of agents to learn is independent of pollution, the environmental care does not influence the optimal growth rate at the steady-state.*

Proof. See equation (25). ■

The optimal growth rate, along the BGP, does not depend on the long-run flow of pollution and on the environmental care ζ because the central planner internalizes the turnover of generation and therefore nullifies the impact of the pollution on the balanced growth rate found in the decentralized economy (see appendix F).

The optimal environmental tax implemented by the government to obtain the optimal net flow of pollution is:

$$\vartheta^{op}(t) = K(t)\frac{(\zeta\gamma)^{1+\gamma}}{\gamma}x(t)^{1+\gamma}$$

Like in Hettich (1998), it evolves through time as a same rate than the physical capital (the source of pollution) and it positively depends on environmental care (ζ) and the aggregate consumption to physical capital ratio x. Denoting τ^{op}, the optimal environmental tax normalized by the physical capital stock, we obtain that when the govenment chooses τ^{op} equal to

$$\tau^{op}|_{BGP} = \frac{(\zeta\gamma)^{1+\gamma}}{\gamma}\left(\frac{(1-\alpha)\left[B-(1-\eta)\lambda\right]+\alpha\varrho}{\alpha-(1-\alpha)\zeta\gamma}\right)^{1+\gamma} \tag{26}$$

along the BGP, the net flow of pollution is at its optimal level.

The optimal environmental tax along the BGP is positively influenced by the environmental concern of individuals (ζ), as expected. It is also positively affected by the efficiency of the time invested in education (B) and the subjective time preference (ϱ). Finally, the optimal environmental tax diminishes with the probability of death (λ): the ageing of population (lower λ) leads to a higher optimal environmental tax because older people are the one who accumulated more physical capital (the source of polution) than younger people.

One should make observe that, when liftime is infinite ($\lambda = 0$), that is when there is a single representative agent, the optimal environmental tax $\tau^{op}|_{BGP}$ is sufficient as the only policy instrument to maximize welfare.

7 Conclusion

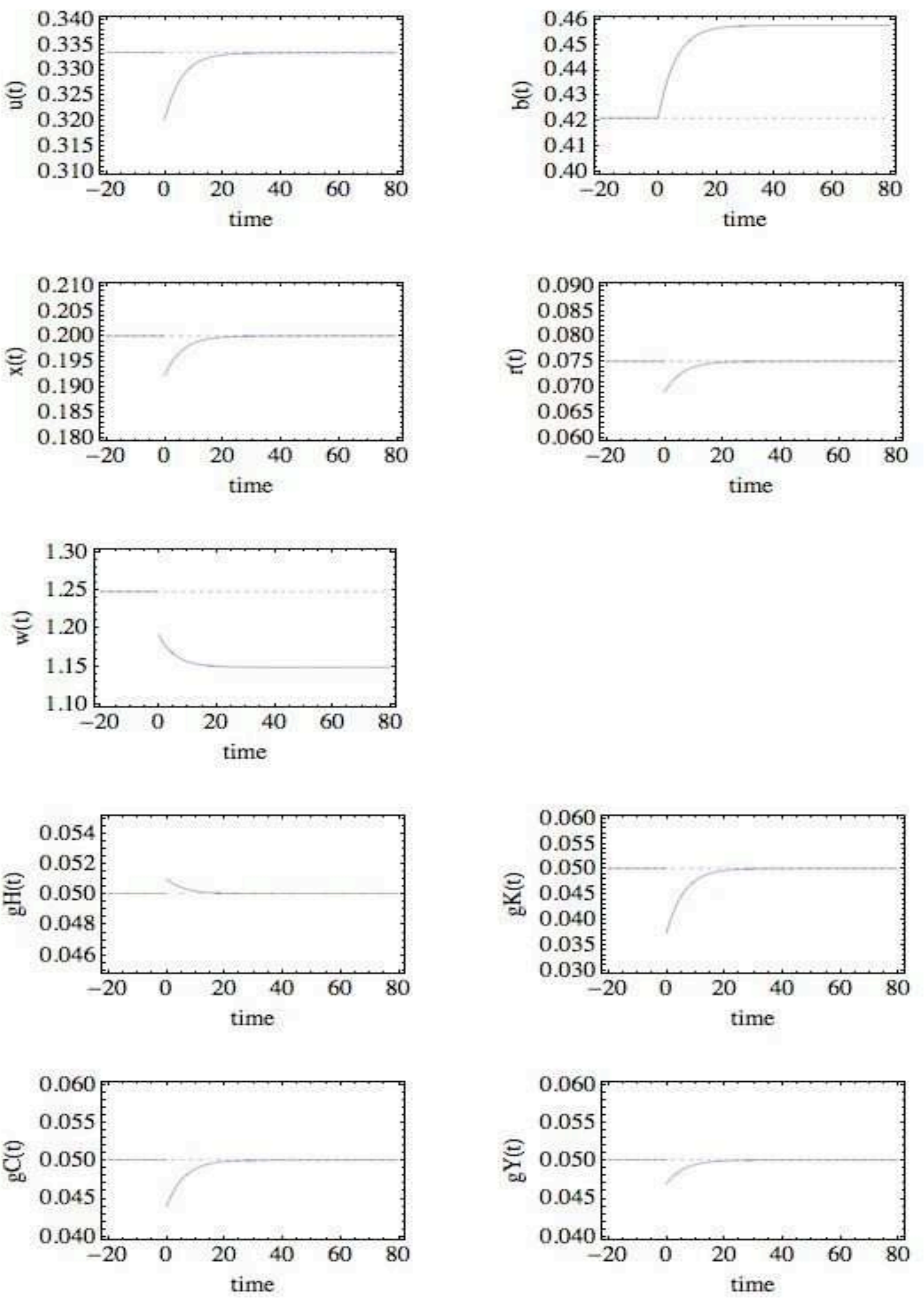
This article demonstrates that, if finite lifetime is taken into account, a win-win environmental policy may be implemented in an economy where growth is driven by human capital accumulation à la Lucas (1988) and the source of pollution is physical capital, while pollution does not influence educational activities, labor supply is not elastic and human capital does not enter the utility function. This is because finite lifetime and the appearance of newborns at each date creates a turnover of generations which disconnects the aggregate consumption growth to the after-tax interest rate. We show that, in the presence of finite lifetime, the ageing of the population (a lower probability to die) reduces the positive influence of the environmental policy on growth.

We also demonstrate that when time is not the single production factor in education, the environmental policy promotes growth only if time remains the predominant factor. Otherwise, the crowding-out effect of the tighter environmental policy dominates the "*generational turnover effect*" and growth diminishes.

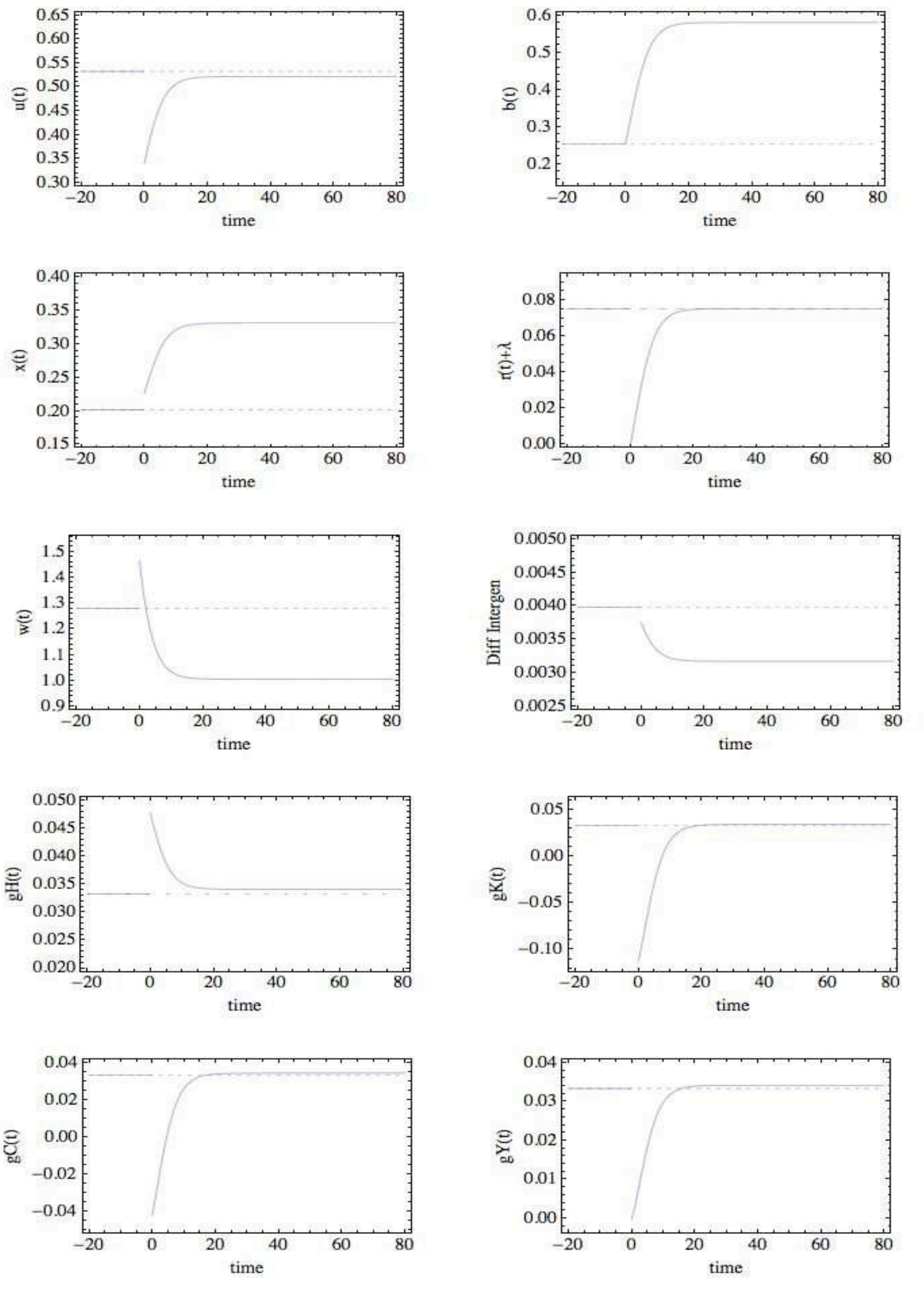
When the source of pollution is final output rather than physical capital and time is the single factor in education, we demonstrate that the environmental tax does not affect growth in the steady-state, despite the "*generational turnover effect*" . Nevertheless, if the education good is introduced, the negative influence of the environmental policy appears because the education good is crowded-out by the tighter environmental policy.

Finally, we demonstrate that BGP optimal growth rate is independent from the environmental care despite the finite lifetime. This is because the central planner internalizes the "*generational turnover effect*" . Such a result is similar to the one found by Vellinga (1999) with infinite lifetime and a representative agent that environmental care does not influence optimal growth when utility is additive and pollution does not influence the ability of agents to be educated.

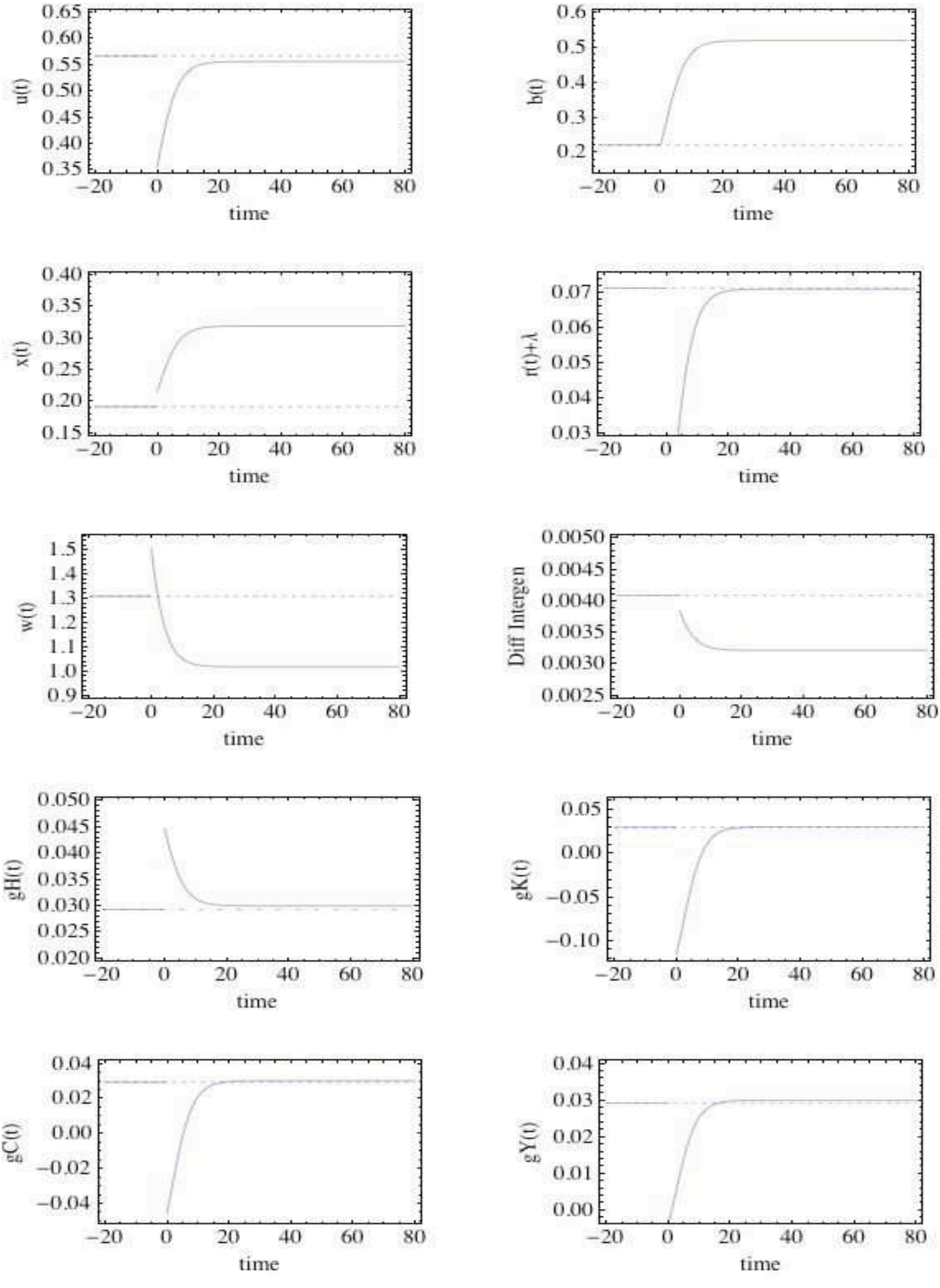
Graph 1. Infinite lifetime (*source of pollution K*)



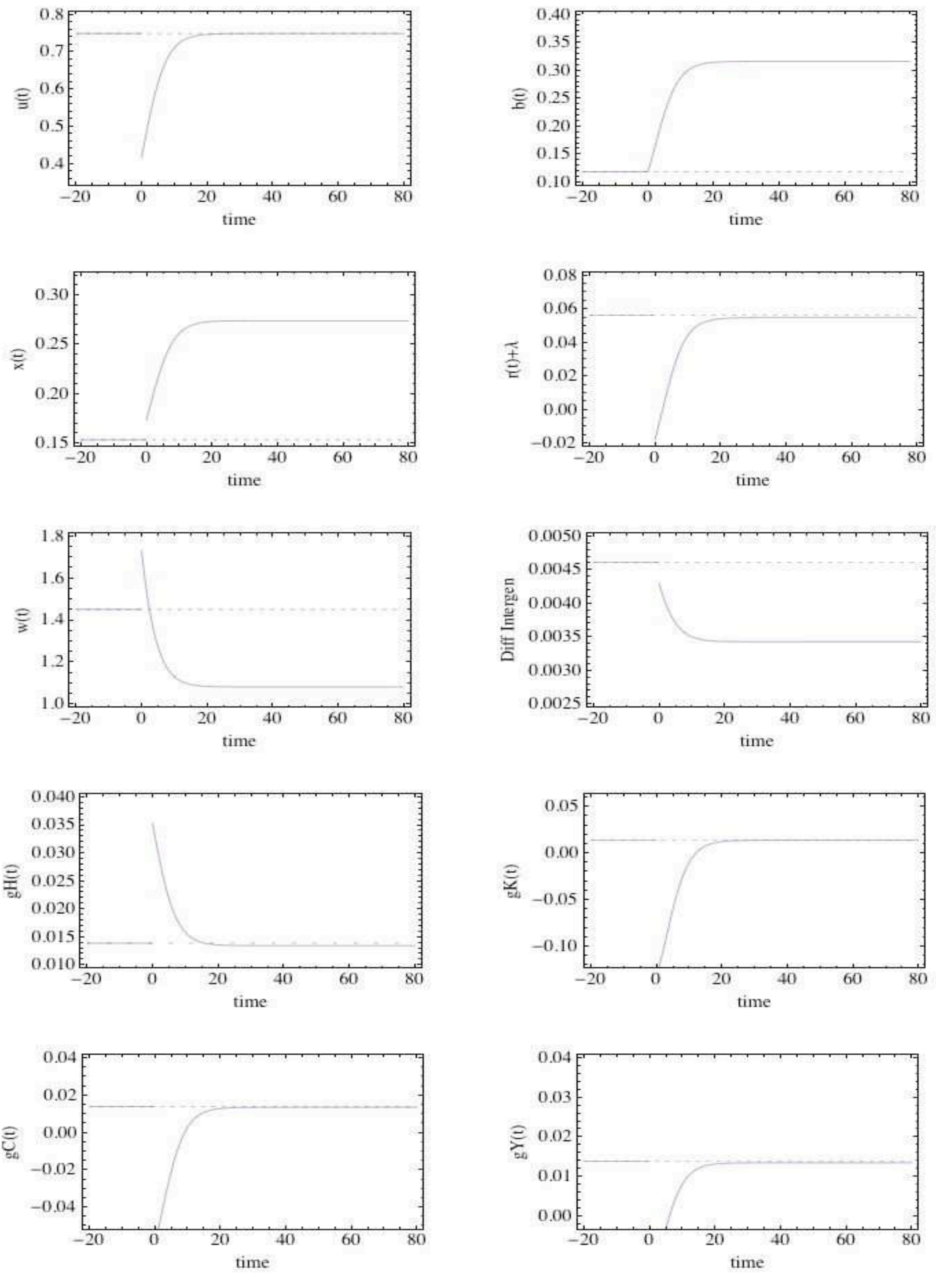
Graph 2. Finite lifetime (*source of pollution K*, δ=0)



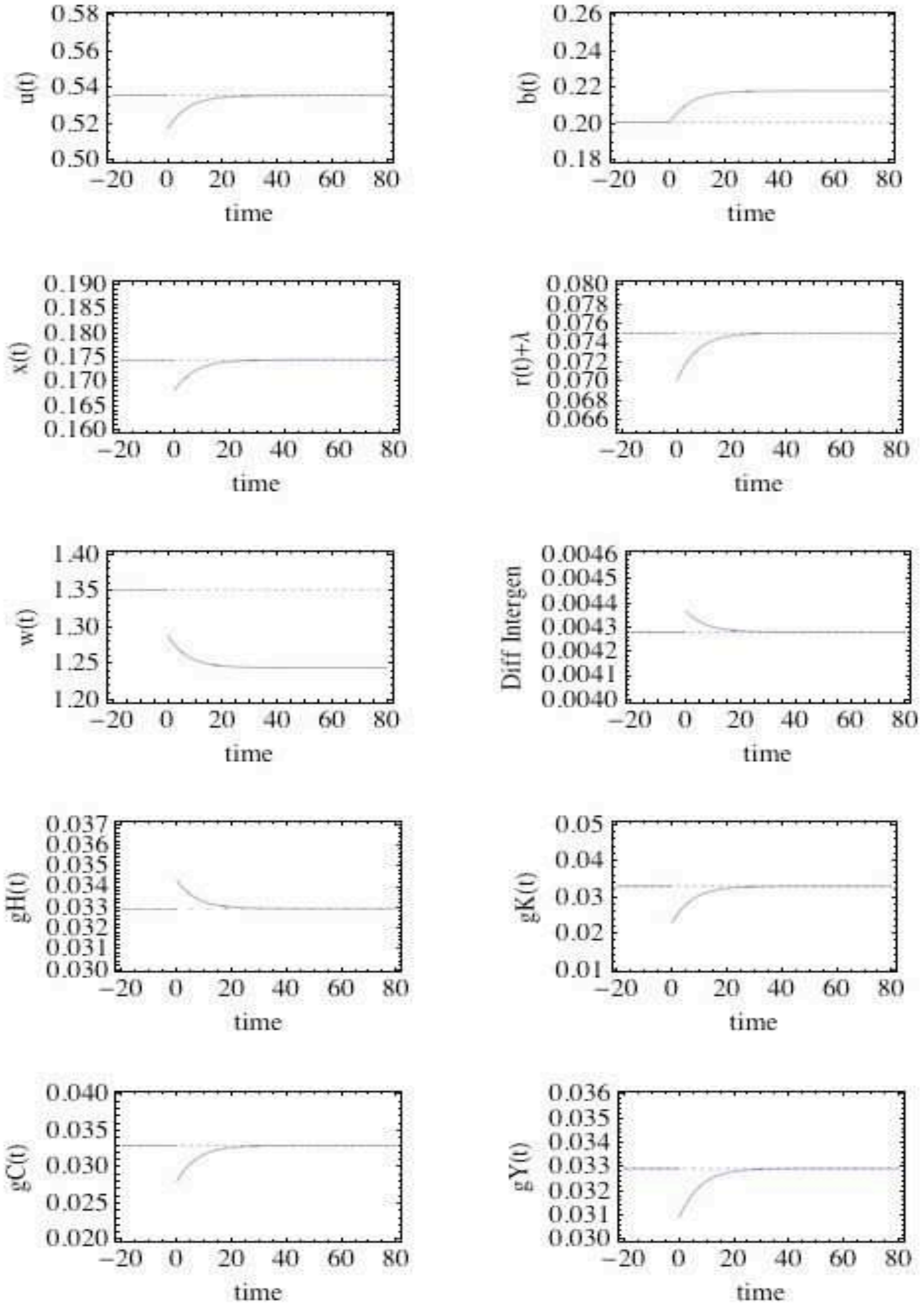
Graph 3. Finite lifetime (*source of pollution K* ; δ=0.01)



Graph 4. Finite lifetime (*source of pollution K* ; δ=0.1)



Graph 5. Finite lifetime (*source of pollution Y* ; δ=0)



References

Barro, R. and Sala-I-Martin, X. (1995). *Economic Growth.* MacGraw-Hill, New-York.

Blanchard, O. (1985). Debt, deficits and finite horizon. *Journal of Political Economy*, 93:223–247.

Blanchard, O. and Fisher, S. (1989). *Lectures on Macroeconomics.* MIT Press, cambridge, MA.

Bovenberg, A. L. and de Mooij, R. A. (1997). Environmental tax reform and endogenous growth. *Journal of Public Economics*, 63(2):207–237.

Bovenberg, A. L. and Heijdra, B. J. (1998). Environmental tax policy and intergenerational distribution. *Journal of Public Economics*, 67(1):1–24.

Bovenberg, A. L. and Heijdra, B. J. (2002). Environmental abatement and intergenerational distribution. *Environmental and Resource Economics*, 23(1):45–84.

Bovenberg, A. L. and Smulders, S. (1995). Environmental quality and pollution-augmenting technological change in a two-sector endogenous growth model. *Journal of Public Economics*, 57(3):369–391.

Bovenberg, A. L. and Smulders, S. A. (1996). Transitional impacts of environmental policy in an endogenous growth model. *International Economic Review*, 37(4):861–893.

Brock, W. and Taylor, M. (2005). Economic growth and the environment: A review of theory and empirics. In *Handbook of Economic Growth, Vol.1, part 2*, pages 1749–1821. Elsevier, North-Holland.

Calvo, G. and Obstfeld, M. (1988). Optimal time-consistent fiscal policy with finite lifetimes. *Econometrica*, 56(2):411–432.

Glomm, G. and Ravikumar, B. (2001). Human capital accumulation and endogenous public expenditures. *Canadian Journal of Economics*, 34(3):807–826.

Gradus, R. and Smulders, S. (1993). The trade-off between environmental care and long-term growth-pollution in three prototype growth models. *Journal of Economics*, 58(1):25–51.

Grimaud, A. and Rouge, L. (2005). Polluting non-renewable resources, innovation and growth: Welfare and environmental policy. *Resource and Energy Economics*, 27(2):109–129.

Grimaud, A. and Rouge, L. (2008). Environment, directed technical change and economic policy. *Environmental and Resource Economics*, 41(4):439–463.

Grimaud, A. and Tournemaine, F. (2007). Why can environmental policy tax promote growth through the channel of education? *Ecological Economics*, 62(1):27–36.

Guvenen, F. (2006). Reconciling conflicting evidence on the elasticity of intertemporal substitution: A macroeconomic perspective. *Journal of Monetary Economics*, 53:1451–1472.

Heijdra, B. and Ligthart, J. (2000). The dynamic macroeconomic effects of tax policy in an overlapping generations model. *Oxford Economic Papers*, 52:677–701.

Heijdra, B. and van der Ploeg, F. (2002). *Foundations of Modern Macroeconomics*. Oxford University Press.

Hettich, F. (1998). Growth effects of a revenue-neutral environmental tax reform. *Journal of Economics*, 67(3):287–316.

King, R. and Rebelo, S. (1990). Public policy and economic growth : Developing neoclassical implications. *Journal of Political Economy*, 98(5):S126–S151.

Lucas, R. (1988). On the mechanisms of economic development. *Journal of Monetary Economics*, 22:3–42.

Ono, T. (2002). The effects of emission permits on growth and the environment. *Environmental and Resource Economics*, 21(1):75–87.

Ono, T. (2003). Environmental tax policy in a model of growth cycles. *Economic Theory*, 22(1):141–168.

Rebelo, S. (1991). Long-run policy analysis and long-run growth. *Journal of Political Economy*, 99:500–521.

Schou, P. (2000). Polluting non-renewable resources and growth. *Environmental and Resource Economics*, 16(2):211–227.

Schou, P. (2002). When environmental policy is superfluous: Growth and polluting resources. *Scandinavian Journal of Economics*, 104(4):605–620.

Song, E. (2002). Taxation, human capital and growth. *Journal of Economic Dynamics and Control*, 26:205–216.

van Ewijk, C. and van Wijnbergen, S. (1995). Can abatement overcome the conflict between environment and economic growth? *De Economist*, 143(2):197–216.

Vellinga, N. (1999). Multiplicative utility and the influence of environmental care on the short-term economic growth rate. *Economic Modelling*, 16(3):307–330.

World Bank (1993). *East-Asian Miracle : Economic Growth and Public Policy*. World Bank, New-York.

Xepapadeas, A. (2005). Economic growth and the environment. In *Handbook of Environmental Economics, vol.III*, pages 1219–1271. Elsevier Science Publishers B.V.

Yaari, M. (1965). Uncertain lifetime, life insurance and the theory of consumer. *Review of Economic Studies*, 32:137–150.

Yogo, M. (2004). Estimating the elasticity of intertemporal substitution when instruments are weak. *The Review of Economics and Statistics*, 86(3):797–810.

Remark: In the following appendix, references to equations without the prefix A are references to the equations in the body of the paper.

Appendix A

This appendix demonstrates how the growth rate of the aggregate consumption in equation (16) is obtained.

Because the individual accumulation of human capital at time ζ by an agent born at $s \leq \zeta$ is $g_h(s,\zeta) \equiv \dot{h}(s,\zeta)/h(s,\zeta) = B[1-u(s,\zeta)]$ (from equation 2), the level of human capital of this agent at time $\nu \geq 0$ is

$$h(s,\nu) = h(s,s)\exp\left[\int_s^\nu g_h(s,\zeta)d\zeta\right]$$

We assumed that when an agent born at time s, he inherits a part $\eta \in]0,1]$ of the aggregate level of human capital $H(s)$:

$$h(s,s) \equiv \eta H(s)$$

and the aggregate human capital evolves such that

$$H(s) = H(0)\exp\left[\int_0^s g_H(s,\zeta)d\zeta\right]$$

with

$$g_H(s,\zeta) \equiv \dot{H}(s,\zeta)/H(s,\zeta) = g_h(s,\zeta) - (1-\eta)\lambda$$

is the growth rate of the aggregate human capital. Recalling that $u(s,\zeta) = u(\zeta)$, that is $g_H(s,\zeta) = g_H(\zeta)$ and $g_h(s,\zeta) = g_h(\zeta)$, we obtain

$$h(s,\nu) = \eta H(0)\exp\left[\left(\int_0^\nu g_h(\zeta)d\zeta\right) - (1-\eta)\lambda \times s\right]$$

As a result, the present value of lifetime earning

$$\omega(s,t) \equiv \int_t^\infty [u(s,\nu)h(s,\nu)w(\nu)]\exp\left[-\int_t^\nu [r(\zeta)+\lambda]d\zeta\right]d\nu$$

may be expressed as

$$\begin{aligned} \omega(s,t) = \quad & \eta H(0)e^{-(1-\eta)\lambda s}\hat{\omega}(t) \\ & \text{with} \\ & \hat{\omega}(t) \equiv \textstyle\int_t^\infty u(\nu)w(\nu)\exp\left[-\int_t^\nu [r(\zeta)+\lambda]d\zeta + \int_0^\nu g_h(\zeta)d\zeta\right]d\nu \end{aligned} \tag{A.1}$$

As a consequence, the aggregate human wealth in the economy $\Omega(t) \equiv \int_{-\infty}^{t} \omega(s,t)\lambda e^{-\lambda(t-s)}ds$ may be expressed as

$$\begin{aligned}\Omega(t) = & \quad \eta H(0)\hat{\omega}(t) \int_{-\infty}^{t} e^{-(1-\eta)\lambda s}\lambda e^{-\lambda(t-s)}ds \\ = & \quad \eta H(0)\hat{\omega}(t) \int_{-\infty}^{t} \lambda e^{\lambda\eta s-\lambda t}ds \\ = & \quad \lambda\eta H(0)\hat{\omega}(t)\left[\frac{e^{\lambda\eta s-\lambda t}}{\lambda\eta}\right]_{-\infty}^{t}\end{aligned}$$

that is

$$\Omega(t) = H(0)e^{-(1-\eta)\lambda t}\hat{\omega}(t)$$

From equation (A.1) (with $s = t$), we obtain that

$$\omega(t,t) = \eta H(0)e^{-(1-\eta)\lambda t}\hat{\omega}(t) = \eta\Omega(t) \tag{A.2}$$

Differentiating (7) with respect to time gives

$$\frac{\dot{C}(t)}{C(t)} = \frac{\dot{c}(s,t)}{c(s,t)} - \frac{1}{C(t)}\left[\lambda C(t) - \lambda c(t,t)\right]$$

that is, using (4)

$$\dot{C}(t) = \left[r(t) - \varrho - \lambda\right]C(t) + \lambda c(t,t)$$

From equation (5) and because agents born without non-human assets ($a(t,t) = 0$), we obtain

$$\dot{C}(t) = \left[r(t) - \varrho - \lambda\right]C(t) + \lambda(\varrho + \lambda)\omega(t,t)$$

Using (A.2) and (7), we find

$$\dot{C}(t) = \left[r(t) - \varrho - \lambda\right]C(t) + \lambda(\varrho + \lambda)\eta\left[\frac{C(t)}{\varrho + \lambda} - K(t)\right]$$

that is

$$\dot{C}(t) = \left[r(t) - \varrho - (1-\eta)\lambda\right]C(t) - \eta\lambda(\varrho + \lambda)K(t) \tag{16}$$

Appendix B

In this Appendix, we solve the "*general case*" where education originates from time-investment and where at time t, each agent born at s buy $z(s,t)$ units of final output which increase the productivity of the time that they invest in education, such as:

$$\dot{h}(s,t) = B(1-u(s,t))^{1-\delta}h(s,t)^{1-\delta}z(s,t)^{\delta}, \qquad \text{with } \delta \in [0,1[\tag{19}$$

When $\delta = 0$, we obtain the Lucas (1988) model with the human capital accumulation given by equation (2).

In the "*general case*" the program of the households is:

$$\begin{array}{rl}\max\limits_{c(s,t),z(s,t),a(s,t),h(s,t),u(s,t)} & \int_s^\infty \left[\log c(s,t) - \zeta \log \mathcal{P}(t)\right] e^{-(\varrho+\lambda)(t-s)} dt \\ s.t. & \dot{a}(s,t) = \left[r(t)+\lambda\right] a(s,t) + u(s,t)h(s,t)w(t) - c(s,t) - z(s,t) \\ & \dot{h}(s,t) = B(1-u(s,t))^{1-\delta} h(s,t)^{1-\delta} z(s,t)^{\delta} \\ & a(s,s,) = 0 \qquad h(s,s) = \eta H(s) > 0 \end{array}$$

The Hamiltonian of the program may be written as:

$$\begin{aligned}\mathcal{H} = \left[\log c(s,t) - \zeta \log \mathcal{P}(t)\right] + \pi_1(t)\left[(r(t)+\lambda)a(s,t) + u(s,t)h(s,t)w(t) - c(s,t) - z(s,t)\right] \\ + \pi_2(t)B(1-u(s,t))^{1-\delta}h(s,t)^{1-\delta}z(s,t)^{\delta}\end{aligned}$$

The F.O.C. are

$$\frac{\partial \mathcal{H}}{\partial c(s,t)} = 0 \qquad \Rightarrow \qquad \frac{1}{c(s,t)} = \pi_1(t) \tag{A.3}$$

$$\frac{\partial \mathcal{H}}{\partial z(s,t)} = 0 \qquad \Rightarrow \qquad \pi_1(t) = \pi_2(t)\delta B(1-u(s,t))^{1-\delta}\left(\frac{z(s,t)}{h(s,t)}\right)^{\delta-1} \tag{A.4}$$

$$\frac{\partial \mathcal{H}}{\partial u(s,t)} = 0 \qquad \Rightarrow \qquad \pi_1(t)w(t) = \pi_2(t)B(1-\delta)(1-u(s,t))^{-\delta}\left(\frac{z(s,t)}{h(s,t)}\right)^{\delta} \tag{A.5}$$

$$\frac{\partial \mathcal{H}}{\partial a(s,t)} = -\dot{\pi}_1(t) + (\varrho+\lambda)\pi_1(t) \qquad \Rightarrow \qquad \pi_1(t)(r(t)+\lambda) = -\dot{\pi}_1(t) + (\varrho+\lambda)\pi_1(t) \tag{A.6}$$

$$\begin{aligned}&\frac{\partial \mathcal{H}}{\partial h(s,t)} = -\dot{\pi}_2(t) + (\varrho+\lambda)\pi_2(t) \qquad \Rightarrow \\ &\quad \pi_1(t)w(t)u(s,t) + \pi_2(t)B(1-\delta)(1-u(s,t))^{1-\delta}\left(\frac{z(s,t)}{h(s,t)}\right)^{\delta} = -\dot{\pi}_2(t) + (\varrho+\lambda)\pi_2(t)\end{aligned} \tag{A.7}$$

First of all, equation (A.5) implies that the ratio $\frac{z(s,t)}{(1-u(s,t))h(s,t)}$ is independent from s and equation (A.7) implies that $u(s,t)$ is independent from s. Consequently, the ratio $\frac{z(s,t)}{h(s,t)}$ is independent from s. Conveniently, we denote $\tilde{z}(t) \equiv \frac{z(s,t)}{(1-u(s,t))h(s,t)}$.

From (A.3) and (A.6), we obtain

$$\dot{c}(s,t) = (r(t)-\varrho)c(s,t) \tag{A.8}$$

Equations (A.4) and (A.5) give:

$$\tilde{z}(t) = \frac{\delta}{1-\delta}w(t) \tag{20}$$

And from equations (A.4) and (A.7):

$$\frac{\dot{\pi}_2(t)}{\pi_2(t)} = \varrho + \lambda - B\left[1-\delta\right]\tilde{z}(t)^{\delta}$$

Differentiating (A.5) with respect to time, we obtain:

$$\frac{\dot{\pi}_1(t)}{\pi_1(t)} + \frac{\dot{w}(t)}{w(t)} = \frac{\dot{\pi}_2(t)}{\pi_2(t)} + \delta\frac{\dot{\tilde{z}}(t)}{\tilde{z}(t)}$$

Replacing by the expressions of $\frac{\dot{\pi}_1(t)}{\pi_1(t)}$ and $\frac{\dot{\pi}_2(t)}{\pi_2(t)}$, it gives

$$\frac{\dot{w}(t)}{w(t)} - \delta\frac{\dot{\tilde{z}}(t)}{\tilde{z}(t)} + B(1-\delta)\tilde{z}(t)^{\delta} = r(t) + \lambda$$

which means that the returns to education must be equal to the returns to physical capital.

We can re-write this relation in terms of $w(t)$ and $r(t)$:

$$(1-\delta)\frac{\dot{w}(t)}{w(t)} + B(1-\delta)^{1-\delta}\delta^{\delta}w(t)^{\delta} = r(t) + \lambda \tag{A.9}$$

From equations (10) and (20), it is possible to express $\tilde{z}(t)$ in terms of $Y(t)$:

$$\tilde{z}(t) = \frac{\delta}{1-\delta}(1-\alpha)\frac{Y(t)}{u(t)H(t)}$$

Now, we can write that the amount of final output used as educational good is $Z(t) \equiv \int_{-\infty}^{t} z(s,t)\lambda e^{-\lambda(t-s)}ds = \int_{-\infty}^{t}\left(\frac{z(s,t)}{(1-u(s,t))h(s,t)}\right)(1-u(s,t))h(s,t)\lambda e^{-\lambda(t-s)}ds = \tilde{z}(t)(1-u(t))\int_{-\infty}^{t} h(s,t)\lambda e^{-\lambda(t-s)}ds = \tilde{z}(t)(1-u(t))H(t)$.

Consequently:

$$Z(t) = \frac{\delta(1-\alpha)}{1-\delta}\left(\frac{1-u(t)}{u(t)}\right)Y(t),$$

the market clearing condition is written as:

$$\left(1 + \frac{(1-\alpha)\delta}{1-\delta}\left(1 - \frac{1}{u(t)}\right)\right)Y(t) = \dot{K}(t) + C(t) + \Phi(\tau)K(t)$$

and the aggregate accumulation of human capital is:

$$\dot{H}(t) = \left[B(1-u(t))\tilde{z}(t)^{\delta} - (1-\eta)\lambda\right]H(t)$$

Finally, from equations (A.9), (10) and the fact that $\int_{-\infty}^{t} u(s,t)h(s,t)\lambda e^{-\lambda(t-s)}ds = u(t)H(t)$ because $u(s,t) = u(t)$, we obtain

$$\frac{\dot{u}(t)}{u(t)} = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{H}(t)}{H(t)} - \alpha^{-1}\left[r(t) + \lambda - B(1-\delta)^{1-\delta}\delta^{\delta}(b(t)u(t))^{-\alpha\delta}\right]$$

Therefore, the dynamical system is summarized by

$$\begin{aligned}
\dot{x}(t) = & \left\{\left[\alpha - \left(1 - \Delta\left(\frac{1}{u(t)} - 1\right)\right)\right](b(t)u(t))^{1-\alpha} - \varrho - (1-\eta)\lambda - \eta\lambda(\varrho+\lambda)x(t)^{-1} + x(t)\right\} x(t) \\
\dot{b}(t) = & \left\{(1-u(t))B\Delta^{\delta}(b(t)u(t))^{-\alpha\delta}\right. \\
& \left. -(1-\eta)\lambda + x(t) + \Phi(\tau) - \left(1 - \Delta\left(\frac{1}{u(t)} - 1\right)\right)(b(t)u(t))^{1-\alpha}\right\} b(t) \\
\dot{u}(t) = & \left\{\left[\alpha^{-1} - 1 + u(t)\right] B\Delta^{\delta}(b(t)u(t))^{-\alpha\delta} + \left((\alpha(1-\delta))^{-1} - 1\right)\Phi(\tau)\right. \\
& \left. -\Delta\left[\left(\frac{1}{u(t)} - 1\right) + \frac{1}{1-\alpha}\right](b(t)u(t))^{1-\alpha} - \left((\alpha(1-\delta))^{-1} + \eta - 1\right)\lambda - x(t)\right\} u(t)
\end{aligned} \tag{A.10}$$

with $\Delta \equiv \frac{(1-\alpha)\delta}{1-\delta}$.

The two last equations of the dynamical system (A.10) evaluated in the steady-state ($\dot{b}(t) = 0$ and $\dot{u}(t) = 0$) gives $b^\star u^\star$ as the solution of the following equality

$$(1-\delta)B\Delta^{\delta}(b^\star u^\star)^{-\alpha\delta} = \alpha(b^\star u^\star)^{1-\alpha} - \Phi(\tau) + \lambda \tag{21}$$

where the left-hand side is the returns to human capital accumulation along the BGP and the right-hand side is the effective interest rate (the returns to physical capital accumulation, see equation (A.9)),[19] both evaluated along the BGP.

When $\delta \in]0,1[$, the left-hand side is a decreasing function of $b^\star u^\star \in]0,+\infty[$ with $\lim_{b^\star u^\star \to 0} LHS = +\infty$ and $\lim_{b^\star u^\star \to +\infty} LHS = 0$, and the right-hand side is an increasing function of $b^\star u^\star$ with $\lim_{b^\star u^\star \to 0} RHS = \lambda - \Phi(\tau)$ and $\lim_{b^\star u^\star \to +\infty} RHS = +\infty$. Consequently, the equality (21) defines a unique $b^\star u^\star \in]0,+\infty[$. Because $\delta \in]0,1[$ and $\Phi'(\tau) > 0$, it is straightforward using the theorem of the implict function that $b^\star u^\star$ is an increasing function of τ and B. When $\delta = 0$, equation (21) becomes $B = \alpha(b^\star u^\star)^{1-\alpha} - \Phi(\tau) + \lambda$ and defines an explicit expression for $b^\star u^\star$:

$$b^\star u^\star = \left(\frac{B - \lambda + \Phi(\tau)}{\alpha}\right)^{1/(1-\alpha)}$$

$b^\star u^\star$ is always an increasing function of B and τ and is positive under the sufficient condition that the returns to education B corrected by the probability of death λ is positive:

$$B - \lambda > 0 \tag{A.11}$$

For convenience, we denote $\mathcal{R}(B,\tau)$, the solution $b^\star u^\star$ of equality (21), with $d\mathcal{R}(B,\tau)/d\tau > 0$ and $d\mathcal{R}(B,\tau)/dB > 0$ for $\delta \in [0,1[$.

[19]See footnote 8 page 7, for the definition of the effective interest rate.

Along the BGP, $\dot{x} = \dot{b} = 0$ defines $x^\star$ as follows:

$$x^\star = \frac{\eta\lambda(\varrho+\lambda)}{\alpha\mathcal{R}(B,\tau)^{1-\alpha} - \Phi(\tau) - \varrho - (1-u^\star)B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta}} \tag{A.12}$$

Using equation (21) we can re-write (from the previous expression) $x^\star$ as a function $\mathcal{X}_1(u^\star,\tau)$:

$$x^\star = \mathcal{X}_1(u^\star,\tau) \equiv \frac{\eta\lambda(\varrho+\lambda)}{(u^\star-\delta)B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta} - \lambda - \varrho}$$

To obtain $x^\star > 0$, we impose that

$$(u^\star-\delta)B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta} > \varrho + \lambda \tag{A.13}$$

ensuring that human capital will not be fully invested in human capital accumulation along the balanced growth path:[20]

$$u^\star > \underline{u}_\delta, \qquad \text{with} \qquad \underline{u}_\delta \equiv \delta + \frac{\varrho+\lambda}{B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta}} \in]0,1[\tag{C1}$$

This condition ensures that the growth rate of human capital does not exceed the maximum feasible rate (when the total amount of human capital is allocated to education).

We also assume that the balanced growth path rate of growth $g^\star$ must be positive, that is (from equation 14):

$$(1-u^\star)B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta} > (1-\eta)\lambda \tag{A.14}$$

This assumption imposes that the investment in education is positive (from equation 14):

$$u^\star < \bar{u}_\delta, \qquad \text{with} \qquad \bar{u}_\delta \equiv 1 - \frac{(1-\eta)\lambda}{B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta}} \in]0,1[\tag{C2}$$

Under conditions (C1)-(C2) the following inequality holds (by summing (A.13) and (A.14)):

$$(1-\delta)B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta} > \varrho + (2-\eta)\lambda, \qquad \text{for } \delta \in [0,1[\tag{A.15}$$

and enables us to demonstrate that $\underline{u}_\delta < \bar{u}_\delta$.

Because $\eta \in]0,1]$, conditions (C1)-(C2) impose $(1-\delta)B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta} > \varrho+\lambda > (1-\eta)\lambda$ (from the inequality (A.15)), that is they imply a positive growth rate of individual consumption $c(s,t)$ (see equations A.8 and 21). Finally, condition (A.11) is verified under conditions (C1)-(C2) (from the inequality (A.15)). And $\lim_{u^\star\to\underline{u}_\delta} \chi_1(u^\star;\tau) = +\infty$ and $\lim_{u^\star\to\bar{u}_\delta} \chi_1(u^\star;\tau) = \frac{\eta\lambda(\varrho+\lambda)}{(1-\delta)B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta}-(2-\eta)\lambda-\varrho} > 0$.

[20] In the Lucas (1988)'s model (see page 23), the maximum feasible rate is B (with our notation), that is $g^\star < B \Leftrightarrow u^\star > 0$. Here, conditions (C1)-(C2) ensure that the BGP growth rate of individual consumption $\dot{c}(s,t)/c(s,t)|_{BGP}$ is higher than the individual accumulation of human capital $\dot{h}(s,t)/h(s,t)|_{BGP}$, that is $c(s,t)|_{BGP} > 0$ (see the denominator of equation A.12).

From $\dot{u}(t)=0$, we can also define $x^\star$ as follows:

$$x^\star = \left[\alpha^{-1}-1+u^\star\right] B\Delta^\delta \mathcal{R}(B,\tau)^{-\alpha\delta} + \mathcal{A}_\delta \Phi(\tau) - \Delta\left(\frac{\alpha}{1-\alpha}+\frac{1}{u^\star}\right)\mathcal{R}(B,\tau)^{1-\alpha} - \left(\mathcal{A}_\delta+\eta\right)\lambda \quad \text{(A.16)}$$

with $\Delta \equiv \frac{(1-\alpha)\delta}{1-\delta}$ and $\mathcal{A}_\delta \equiv (\alpha(1-\delta))^{-1}-1>0$, $\forall \delta \in [0,1[$. Using (21) and simplifying, we can express $x^\star$ as a function $\mathcal{X}_2(u^\star,\tau)$:

$$x^\star = \mathcal{X}_2(u^\star,\tau) \equiv (u^\star-\delta)B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta} - \eta\lambda + \frac{(1-\alpha)(u^\star-\delta)}{(1-\delta)u^\star}\mathcal{R}(B,\tau)^{1-\alpha} \quad \text{(A.17)}$$

It is straightforward that $\chi_2(u^\star;\tau)$ is an increasing function of $u^\star$ and $\chi_2(u^\star;\tau)>0$ for all $u^\star \in]\underline{u}_\delta, \bar{u}_\delta[$ and $\delta \in [0,1[$.[21] When $\delta=0$, it is straightforward that $\chi_2(u^\star;\tau)$ is an increasing function of τ, but for $\delta \in]0,1[$, the influence of τ is unclear.

The BGP equilibrium is defined by $\chi_1(u^\star;\tau)=\chi_2(u^\star;\tau)$ for $u^\star \in]\underline{u}_\delta, \bar{u}_\delta[$. That is, there exists, for $\delta \in [0,1[$, a unique $u^\star \in]\underline{u}_\delta, \bar{u}_\delta[$, solution of $\Gamma_\delta(u;\tau)=0$ with

$$\Gamma_\delta(u;\tau) \equiv \left[(u-\delta)B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta}-\lambda-\varrho\right]\times \left\{(u-\delta)B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta}-\eta\lambda+\frac{(1-\alpha)(u^\star-\delta)}{(1-\delta)u}\mathcal{R}(B,\tau)^{1-\alpha}\right\} - \eta\lambda(\varrho+\lambda)$$

and $\mathcal{R}(B,\tau)$ is defined by equation (21).

It is straightforward that, for $u^\star \in]\underline{u}_\delta, \bar{u}_\delta[$ (see conditions (C1)-(C2)), $\Gamma_\delta(\underline{u}_\delta;\tau) = -\eta\lambda(\varrho+\lambda)<0$ and $\Gamma_\delta(\bar{u}_\delta;\tau)>0$.[22] Because in the interval $]\underline{u}_\delta, \bar{u}_\delta[$, $\Gamma_\delta(u;\tau)$ is a monotonic increasing function of u, $u^\star$ solution of $\Gamma_\delta(u;\tau)=0$ is unique. The influence of τ on $u^\star$ is not clear except when $\delta=0$.[23]

Appendix B1. The case $\delta = 0$

In that case, $\Delta=0$ and $\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta}=1$. The system (A.10) simplifies to:

$$\begin{aligned}
\dot{x}(t) &= \left\{[\alpha-1]\,(b(t)u(t))^{1-\alpha}-\varrho-(1-\eta)\lambda-\eta\lambda(\varrho+\lambda)x(t)^{-1}+x(t)\right\}x(t)\\
\dot{b}(t) &= \left\{(1-u(t))B-(1-\eta)\lambda+x(t)+\Phi(\tau)-(b(t)u(t))^{1-\alpha}\right\}b(t)\\
\dot{u}(t) &= \left\{\left[\alpha^{-1}-1+u(t)\right]B+\mathcal{A}_0\Phi(\tau)-\left(\mathcal{A}_0+\eta\right)\lambda-x(t)\right\}u(t)
\end{aligned} \quad (17)$$

with $\mathcal{A}_0 \equiv \alpha^{-1}-1>0$.

[21] See appendix B1 below for a formal proof when $\delta=0$.

[22] Because $\left[(1-\delta)B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta}-(2-\eta)\lambda-\varrho\right]\left[(1-\delta)B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta}-\lambda\right] > \eta\lambda(\varrho+\lambda)$, we obtain $\Gamma_\delta(\bar{u}_\delta;\tau)>0$.

[23] See numerical simulations in the main text for $\delta \neq 0$.

Equation (21) gives the explicit expression of $b^\star u^\star$:

$$b^\star u^\star = \left(\frac{B-\lambda+\Phi(\tau)}{\alpha}\right)^{1/(1-\alpha)}$$

with the condition

$$B > \lambda - \Phi(\tau) \tag{A.18}$$

to ensure that the returns to education is sufficient to obtain its equalization with the effective interest rate.

The existence condition (C1) becomes:

$$u^\star > \underline{u} \qquad \text{with} \qquad \underline{u} \equiv \frac{\varrho+\lambda}{B} \tag{C1$|_{\delta=0}$}$$

and the existence condition (C2) becomes:

$$u^\star < \bar{u} \qquad \text{with} \qquad \bar{u} \equiv 1 - \frac{(1-\eta)\lambda}{B} \tag{C2$|_{\delta=0}$}$$

Under conditions (C1$|_{\delta=0}$)-(C2$|_{\delta=0}$), the inequality (A.15) holds (with $\delta = 0$ and $\Delta^\delta \mathcal{R}(B,\tau)^{-\alpha\delta} = 1$) and therefore $0 < \underline{u} < \bar{u} < 1$. Condition (A.18) is also verified and there exists a unique $u^\star \in]\underline{u}, \bar{u}[$ solution of $\Gamma(u;\tau) = 0$ where $\Gamma(u;\tau)$ is defined as follows:[24]

$$\Gamma(u;\tau) \equiv [Bu - \lambda - \varrho] \times [(\mathcal{A}_0 + u)B + \mathcal{A}_0\Phi(\tau) - (\mathcal{A}_0+\eta)\lambda] - \eta\lambda(\varrho+\lambda) = 0$$

with

$$\Gamma(\underline{u};\tau) = -\eta\lambda(\varrho+\lambda) < 0$$

and

$$\Gamma(\bar{u};\tau) = [B - (2-\eta)\lambda - \varrho] \times \left[\alpha^{-1}(B-\lambda) + \mathcal{A}_0\Phi(\tau)\right] - \eta\lambda(\varrho+\lambda) > 0$$

under conditions (C1$|_{\delta=0}$)-(C2$|_{\delta=0}$). [25]

From the implicit function theorem, the influence of τ on $u^\star$ is given by $u^{\star\prime} = -\frac{\partial\Gamma(u;\tau)/\partial\tau}{\partial\Gamma(u;\tau)/\partial u}$. If we note $\Gamma(u;\tau) = \Gamma_1(u;\tau) \times \Gamma_2(u;\tau) - \eta\lambda(\varrho+\lambda)$, with $\Gamma_1(u;\tau) \equiv Bu^\star - \lambda - \varrho > 0$ and $\Gamma_2(u;\tau) \equiv (\mathcal{A}_0+u)B + \mathcal{A}_0\Phi(\tau) - (\mathcal{A}_0+\eta)\lambda > 0$. Therefore $u^{\star\prime} = \frac{-\mathcal{A}\Phi'(\tau)\Gamma_1(u;\tau)}{B[\Gamma_1(u;\tau)+\Gamma_2(u;\tau)]}$ is negative and $u^\star$ is a decreasing function of τ.

If we note $g^{\star\prime} \equiv dg^\star/d\tau = -Bu^{\star\prime}$, the effect of the environmental policy on growth with respect to the horizon is given by $dg^{\star\prime}/d\lambda$.

Because $g^{\star\prime} = \mathcal{A}\Phi'(\tau)\left[1 + \frac{\Gamma_2(u;\tau)}{\Gamma_1(u;\tau)}\right]^{-1}$, and $\frac{\partial u^\star}{\partial\lambda} = \frac{\Gamma_2(u;\tau)+(\mathcal{A}_0+\eta)\Gamma_1(u;\tau)}{B[\Gamma_1(u;\tau)+\Gamma_2(u;\tau)]} > 0$, and $\frac{\partial\Gamma_1(u;\tau)}{\partial\lambda} = (\alpha^{-1} - 2+\eta)\frac{\Gamma_1(u;\tau)}{\Gamma_1(u;\tau)+\Gamma_2(u;\tau)} > 0$ (under the realistic sufficient condition $\alpha < 1/(2-\eta)$) and $\frac{\partial\Gamma_2(u;\tau)}{\partial\lambda} = -(\alpha^{-1} - 2+\eta)\frac{\Gamma_2(u;\tau)}{\Gamma_1(u;\tau)+\Gamma_2(u;\tau)} < 0$, we obtain that $\frac{\partial\Gamma_2(u;\tau)/\Gamma_1(u;\tau)}{\partial\lambda} = -2(\alpha^{-1}-2+\eta)\frac{\Gamma_2(u;\tau)}{\Gamma_1(u;\tau)} < 0$ that is $\frac{\partial g^{\star\prime}}{\partial\lambda} > 0$.

[24] Note that, under conditions (C1$|_{\delta=0}$), the second term into brackets in the expression between the equality sign is always positive for $u^\star \in]\underline{u}, \bar{u}[$. As a proof: $\left[\mathcal{A}_0 B + \underbrace{\varrho+\lambda}_{B\underline{u}} - \mathcal{A}_0\lambda - \eta\lambda + \mathcal{A}_0\Phi(\tau)\right] > 0$.

[25] We obtain $\Gamma(\bar{u};\tau) > 0$ because $\alpha^{-1}(B-\lambda)[B-(2-\eta)\lambda-\varrho] > \eta\lambda(\varrho+\lambda)$ and $[B-(2-\eta)\lambda-\varrho] \times \mathcal{A}_0\Phi(\tau) > 0$.

Appendix B2. The case $\lambda = 0$

With infinite lifetime ($\lambda = 0$), the dynamical system (A.10) becomes:

$$\dot{x}(t) = \left\{\left[\alpha - \left(1 + \Delta\left(1 - \frac{1}{u(t)}\right)\right)\right](b(t)u(t))^{1-\alpha} - \varrho + x(t)\right\} x(t)$$

$$\dot{b}(t) = \left\{(1 - u(t))B\Delta^{\delta}(b(t)u(t))^{-\alpha\delta} + x(t) + \Phi(\tau) - \left(1 + \Delta\left(1 - \frac{1}{u(t)}\right)\right)(b(t)u(t))^{1-\alpha}\right\} b(t)$$

$$\begin{aligned}\dot{u}(t) = \ & \big\{\left[\alpha^{-1} - 1 + u(t)\right] B\Delta^{\delta}(b(t)u(t))^{-\alpha\delta} + \left((\alpha(1-\delta))^{-1} - 1\right)\Phi(\tau) - x(t) \\ & + \Delta\left[\left(1 - \frac{1}{u(t)}\right) - \frac{1}{1-\alpha}\right](b(t)u(t))^{1-\alpha}\Big\} u(t)\end{aligned}$$

From $\dot{u} = 0$ and $\dot{b} = 0$, we obtain the equality between the returns to investment:

$$(1 - \delta)B\Delta^{\delta}(b^{\star}u^{\star})^{-\alpha\delta} = \alpha(b^{\star}u^{\star})^{1-\alpha} - \Phi(\tau)$$

that defines $b^{\star}u^{\star}$ as an increasing function of τ denoted $\mathcal{R}(B,\tau)\,|_{\lambda=0}$. Using $\dot{x} = \dot{b} = 0$, we obtain the expression of $u^{\star}$:

$$u^{\star} = \delta + \frac{\varrho}{B\Delta^{\delta}\left(\mathcal{R}(B,\tau)\,|_{\lambda=0}\right)^{-\alpha\delta}}$$

which is increasing in τ and the growth rate along the BGP is:

$$g^{\star} = (1 - \delta)B\Delta^{\delta}\left(\mathcal{R}(B,\tau)\,|_{\lambda=0}\right)^{-\alpha\delta} - \varrho$$

Therefore, the environmental tax reduces the human capital accumulation along the BGP when lifetime is infinite. When $\delta = 0$, we find the solution of the Lucas (1988) model with logarithmic utility: $u^{\star} = \varrho/B$ and $g^{\star} = B - \varrho$.

Appendix C

The dynamical system (A.10) may be linearized around the steady-state and becomes:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{b}(t) \\ \dot{u}(t) \end{pmatrix} = \mathcal{J} \times \begin{pmatrix} x(t) - x^{\star} \\ b(t) - b^{\star} \\ u(t) - u^{\star} \end{pmatrix}$$

where $\mathcal{J}$ is the Jacobian matrix evaluated at the neighbourhood of the steady-state:

$$\mathcal{J} \equiv \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{pmatrix}$$

with

$$
\begin{aligned}
J_{11} = &\ \eta\lambda(\varrho+\lambda)x^{\star-1} + x^\star > 0\\
J_{12} = &\ \tfrac{-(1-\alpha)^2(u^\star-\delta)}{(1-\delta)u^\star}\mathcal{R}(B,\tau)^{1-\alpha}x^\star/b^\star < 0\\
J_{13} = &\ -(1-\alpha)\mathcal{R}(B,\tau)^{1-\alpha}\left[\tfrac{1-\alpha}{1-\delta}+\alpha\tfrac{\Delta}{u^\star}\right]x^\star/u^\star < 0\\
J_{21} = &\ b^\star > 0\\
J_{22} = &\ -\alpha\delta(1-u^\star)B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta} - (1-\alpha)\left[1-\Delta\left(\tfrac{1}{u^\star}-1\right)\right]\mathcal{R}(B,\tau)^{1-\alpha} < 0\\
J_{23} = &\ -\left[1+\alpha\delta(1/u^\star-1)\right]B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta}b^\star - (1-\alpha\delta)\left(\tfrac{1-\alpha}{1-\delta}\right)\mathcal{R}(B,\tau)^{1-\alpha}b^\star/u^\star < 0\\
J_{31} = &\ -u^\star < 0\\
J_{32} = &\ -u^\star/b^\star\left\{\alpha\delta(\alpha^{-1}-1+u^\star)B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta} + (1-\alpha)\Delta\left[\tfrac{1}{u^\star}+\tfrac{\alpha}{1-\alpha}\right]\mathcal{R}(B,\tau)^{\star 1-\alpha}\right\} < 0\\
J_{33} = &\ B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta}\left[u^\star-\delta+\alpha\delta(1-u^\star)\right] + \alpha\Delta\left(\tfrac{1}{u^\star}-1\right)\mathcal{R}(B,\tau)^{1-\alpha} > 0
\end{aligned}
$$

The determinant of the Jacobian matrix is

$$
\det(\mathcal{J}) = J_{22}\left(J_{11}J_{33}-J_{13}J_{31}\right) + J_{32}\left(J_{21}J_{13}-J_{23}J_{11}\right) + J_{12}\left(J_{31}J_{23}-J_{33}J_{21}\right) < 0
$$

because, under conditions (C1)-(C2) (inequality (A.15) holds),

$$
\begin{aligned}
J_{11}J_{33}-J_{13}J_{31} = J_{11}\times\Bigg[&\frac{(1-u^\star)(1-\alpha)\alpha\delta}{u^\star(1-\delta)}\mathcal{R}(B,\tau)^{1-\alpha} + (u^\star-\delta+\alpha\delta(1-u^\star))B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta}\Bigg]\\
&-\frac{(1-\alpha)(u^\star-\delta)x^\star}{u^\star(1-\delta)}\mathcal{R}(B,\tau)^{1-\alpha} > 0
\end{aligned}
$$

$$
\begin{aligned}
J_{21}J_{13}-J_{23}J_{11} = &\frac{b^\star}{(1-\delta)u^{\star 2}}\times\Big\{\eta\lambda(\varrho+\lambda)u^\star\Big(\delta(1-\delta)B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta} + (1-\alpha)(1-\alpha\delta)\mathcal{R}(B,\tau)^{1-\alpha}\\
&+(1-\delta)(1-\alpha\delta)u^\star B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta}\Big) + x^\star\Big[\alpha(1-\delta)\Big(\alpha(1-\alpha)(u^\star-(1-\alpha)\delta)\mathcal{R}(B,\tau)^{1-\alpha}\\
&+\delta u^\star x^\star B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta}\Big) + \alpha\delta(1-\alpha)^2(1-\delta)\mathcal{R}(B,\tau)^{1-\alpha}\\
&+(1-\alpha\delta)(1-\delta)u^{\star 2}B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta}\Big]\Big\} > 0
\end{aligned}
$$

$$
J_{31}J_{23}-J_{33}J_{21} = \frac{b^\star}{(1-\delta)u^\star}\left[(1-\alpha)(u^\star-\alpha\delta)\mathcal{R}(B,\tau)^{1-\alpha} + (1-\delta)\delta B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta}\right] > 0
$$

And the trace of the Jacobian matrix is

$$
\mathrm{Trace}(\mathcal{J}) = J_{11}+J_{22}+J_{33} = (u^\star-\delta)\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta} + \eta\lambda(\varrho+\lambda)x^{\star-1} + x^\star - \frac{(1-\alpha)(u^\star-\delta)}{(1-\delta)u^\star}\mathcal{R}(B,\tau)^{1-\alpha}
$$

From equation (A.17) we have $x^\star - \frac{(1-\alpha)(u^\star-\delta)}{(1-\delta)u^\star}\mathcal{R}(B,\tau)^{1-\alpha} = (u^\star-\delta)B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta} - \eta\lambda > 0$, therefore the Trace of the Jacobian matrix is positive.

Because there are two control variables (u and x) and one state-variable (b), the negative determinant and the positive trace of the Jacobian matrix imply that there are two positive eigenvalues and one negative eigenvalue. Therefore, the equilibrium is saddle-path stable.

Note that when $\delta = 0$, we obtain

$$\det(\mathcal{J}) = -(1-\alpha)Bu^\star \left(\mathcal{R}(B,\tau)\right)^{1-\alpha}\left[x^\star + \eta\lambda(\varrho+\lambda)x^{\star-1}\right] < 0$$

and, using equation (A.17) and the inequality (A.15),

$$\mathrm{Trace}(\mathcal{J}) = 2Bu^\star - \eta\lambda + \eta\lambda(\varrho+\lambda)x^{\star-1} > 0$$

Appendix D

In this appendix, we relax the assumption of logarithmic utility to show that this simplifying assumption does not modify the qualitative results. We just give the main equations of model.

The expected utility of an agent born at $s \leq t$ becomes

$$\int_s^\infty \frac{[c(s,t)\mathcal{P}(t)^{-\zeta}]^{1-1/\sigma} - 1}{1-1/\sigma} e^{-(\varrho+\lambda)(t-s)} dt$$

with $\sigma \neq 1$ the (positive) elasticity of intertemporal substitution.

The individual consumption of an agent born at s becomes

$$c(s,t) = \Psi(t)^{-1}[a(s,t) + \omega(s,t)]$$

where $\Psi(t) \equiv \int_t^\infty e^{-(\sigma\varrho+\lambda)(\nu-t)-(1-\sigma)\int_t^\nu r_\mu d\mu} d\nu > 0$ is the propensity to consume out of wealth.[26] Similarly, we can express the aggregate consumption at time t:

$$C(t) = \int_{-\infty}^t c(s,t)\lambda e^{-\lambda(t-s)} ds = \Psi(t)^{-1}\left[K(t) + \Omega(t)\right]$$

The aggregate consumption growth rate (equation 16) becomes

$$\dot{C}(t) = \left[\sigma\left(r(t) - \varrho\right) - (1-\eta)\lambda\right] C(t) - \eta\lambda\Psi(t)^{-1}K(t)$$

Finally, the dynamics of the economy is summarized by the following system (with respect to the case where $\sigma = 1$, the intertemporal evolution of the propensity to consume out of wealth is added):

$$\begin{aligned}
\dot{x}(t) = & \left\{\left[\alpha\sigma - \left(1 - \Delta\left(\frac{1}{u(t)} - 1\right)\right)\right](b(t)u(t))^{1-\alpha} + (1-\sigma)\Phi(\tau) - \sigma\varrho - (1-\eta)\lambda - \frac{\eta\lambda\Psi(t)^{-1}}{x(t)} + x(t)\right\} x(t) \\
\dot{b}(t) = & \left\{(1-u(t))\, B\Delta^\delta(b(t)u(t))^{-\alpha\delta} - (1-\eta)\lambda - \left(1 - \Delta\left(\frac{1}{u(t)} - 1\right)\right)(b(t)u(t))^{1-\alpha} + x(t) + \Phi(\tau)\right\} b(t) \\
\dot{u}(t) = & \Big\{\left(\alpha^{-1} - 1 + u(t)\right) B\Delta^\delta(b(t)u(t))^{-\alpha\delta} \\
& +((\alpha(1-\delta))^{-1} - 1)\Phi(\tau) - \Delta\left[\frac{1}{u(t)} + \frac{\alpha}{1-\alpha}\right] - \left((\alpha(1-\delta))^{-1} + 1 - \eta\right)\lambda - x(t)\Big\} u(t) \\
\dot{\Psi}(t) = & -1 + \left[(1-\sigma)\alpha(b(t)u(t))^{1-\alpha} - (1-\sigma)\Phi(\tau) + \sigma\varrho + \lambda\right]\Psi(t)
\end{aligned}$$

[26] $\Psi(t) > 0$ insures that individual utility is bounded.

where $x(t) \equiv C(t)/K(t)$ and $b(t) \equiv H(t)/K(t)$.

Along the balanced growth path, the propensity to consume out of wealth is constant, that is $\dot{\Psi}(t) = 0$ and as a consequence

$$\Psi^{\star -1} = (1-\sigma)\alpha(b^\star u^\star)^{1-\alpha} - (1-\sigma)\Phi(\tau) + \sigma\varrho + \lambda > 0$$

In the same way than in appendix B, the second and the third equations of the dynamical system evaluated along the BGP ($\dot{u} = \dot{b} = 0$) enable us to express $b^\star u^\star$ as an increasing function of B and τ, denoted $\mathcal{R}(B,\tau)$ and defined by

$$(1-\delta)B\Delta^\delta(b^\star u^\star)^{-\alpha\delta} = \alpha(b^\star u^\star)^{1-\alpha} - \Phi(\tau) + \lambda \tag{21}$$

The two first equations evaluated along the BGP give a first expression of $x^\star$:

$$x^\star = \chi_1^\sigma(u^\star;\tau) \equiv \frac{\lambda\eta\left[(1-\sigma)\alpha\mathcal{R}(B,\tau)^{1-\alpha} - (1-\sigma)\Phi(\tau) + \sigma\varrho + \lambda\right]}{(u^\star - (1-\sigma) - \sigma\delta)B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta} - \sigma(\varrho+\lambda)}$$

Because $x^\star > 0$, we impose a condition similar to condition (C1) (when $\sigma = 1$):

$$u^\star > \underline{\underline{u}}_\delta \qquad \text{with} \qquad \underline{\underline{u}}_\delta \equiv 1 - \sigma + \sigma\left(\delta + \frac{\varrho+\lambda}{B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta}}\right) \tag{C1$|_{\sigma\neq 1}$}$$

and because $\underline{\underline{u}}_\delta \geq 0$, we impose

$$\sigma \leq \left[1 - \left(\delta + \frac{\varrho+\lambda}{B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta}}\right)\right]^{-1}$$

The left-hand side of this inequality being higher than unity, we just impose the sufficient condition $\sigma \leq 1$.[27] This condition is similar to the one defined by Lucas (1988) page 23 (equation 27) according to which: "[...] *the model cannot apply at levels of risk aversion that are too low (that is, if the intertemporal substituability of consumption is too high)*.".[28]

Furthermore, the assumption $g^\star > 0$ always imposes an upper-bound for $u^\star$ defined by equation (C2)

$$u^\star < \bar{u}_\delta \qquad \text{with} \qquad \bar{u}_\delta \equiv 1 - \frac{(1-\eta)\lambda}{B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta}} \tag{C2}$$

with $0 < \underline{\underline{u}}_\delta < \bar{u}_\delta < 1$ because under conditions (C1$|_{\sigma\neq 1}$)-(C2), the inequality (A.15) always holds and we assume $\sigma \leq 1$.

[27]Note that most of the empirical studies about the elasticity of intertemporal substitution of consumption estimate a value lower than unity. For recent references, seeYogo (2004) and Guvenen (2006).

[28]See also Heijdra and van der Ploeg (2002) page 461.

Because the equation $\dot{u}(t)$ is the same than in the case where $\sigma = 1$ (see the system A.10), the second expression of $x^\star$ from $\dot{u} = 0$ is the same than in the case $\sigma = 1$ and is given by

$$x^\star = \mathcal{X}_2(u^\star, \tau) \equiv (u^\star - \delta)B\Delta^\delta \mathcal{R}(B,\tau)^{-\alpha\delta} - \eta\lambda + \frac{(1-\alpha)(u^\star - \delta)}{(1-\delta)u^\star}\mathcal{R}(B,\tau)^{1-\alpha} \quad \text{(A.17)}$$

with $\Delta \equiv \frac{(1-\alpha)\delta}{1-\delta}$ and $\mathcal{A}_\delta \equiv (\alpha(1-\delta))^{-1} - 1 > 0$, $\forall \delta \in [0,1[$.

The BGP equilibrium is defined by $\chi_1^\sigma(u^\star;\tau) = \chi_2(u^\star;\tau)$, under conditions $(\mathrm{C1}|_{\sigma\neq 1})$-(C2) and $\sigma \leq 1$. That is, there exists, for $\delta \in [0,1[$ and $\sigma \leq 1$, a unique $u^\star \in]\underline{\underline{u}}_\delta, \bar{u}_\delta[$, solution of $\Gamma_\delta^\sigma(u;\tau) = 0$ with

$$\begin{aligned}\Gamma_\delta^\sigma(u;\tau) \equiv & \Big[(u - (1-\sigma) - \sigma\delta)B\Delta^\delta \mathcal{R}(B,\tau)^{-\alpha\delta} - \sigma(\varrho + \lambda)\Big] \times \\ & \left\{(u^\star - \delta)B\Delta^\delta \mathcal{R}(B,\tau)^{-\alpha\delta} - \eta\lambda + \frac{(1-\alpha)(u^\star - \delta)}{(1-\delta)u^\star}\mathcal{R}(B,\tau)^{1-\alpha}\right\} \\ & - \eta\lambda\left[(1-\sigma)\alpha\mathcal{R}(B,\tau)^{1-\alpha} - (1-\sigma)\Phi(\tau) + \sigma\varrho + \lambda\right]\end{aligned}$$

It is straightforward that $\Gamma_\delta^\sigma(\underline{\underline{u}}_\delta;\tau) < 0$ (because the first term into brackets in the right-hand side of the equation is null) and $\Gamma_\delta^\sigma(\bar{u}_\delta;\tau) > 0$. Because in the interval $]\underline{\underline{u}}_\delta, \bar{u}_\delta[$, $\Gamma_\delta^\sigma(u;\tau)$ is a monotonic increasing function of u, $u^\star$ solution of $\Gamma_\delta^\sigma(u;\tau) = 0$ is unique.

Note that, when $\lambda = 0$ (infinite lifetime) and $\delta = 0$, we find the result of Lucas (1988) with CARRA preferences: $u^\star = \sigma\varrho/B + 1 - \sigma$ (with $\varrho/B < 1$ to ensure $u^\star < 1$ and $\sigma < [1 - \varrho/B]^{-1}$ to ensure that $u^\star > 0$) and $g^\star = \sigma(B - \varrho) > 0$.

We just investigate the case $\delta = 0$ for simplicity, the general case being cumbersome to study analytically. The condition $(\mathrm{C1}|_{\sigma\neq 1})$ becomes:

$$u^\star > \underline{\underline{u}} \qquad \text{with} \qquad \underline{\underline{u}} \equiv 1 - \sigma\left(1 - \frac{\varrho + \lambda}{B}\right) \qquad (\mathrm{C1}|_{\{\sigma\neq 1, \delta = 0\}})$$

with $\underline{\underline{u}} > 0$ imposes

$$\sigma < \left(1 - \frac{\varrho + \lambda}{B}\right)^{-1}$$

Finally, condition (C2) becomes:

$$u^\star < \bar{u} \qquad \text{with} \qquad \bar{u} \equiv 1 - \frac{(1-\eta)\lambda}{B} \qquad (\mathrm{C2}|_{\delta=0})$$

with $0 < \underline{\underline{u}} < \overline{u} < 1$ because under conditions $(\mathrm{C1}|_{\{\sigma\neq 1,\delta=0\}})$-$(\mathrm{C2}|_{\delta=0})$, the inequality (A.15) always hol (with $\delta = 0$ and $\Delta^\delta \mathcal{R}(B,\tau)^{-\alpha\delta} = 1$) and we assume $\sigma \leq 1$.

Under the condition $\sigma \leq 1$, there exists a unique $u^\star \in]\underline{\underline{u}}, \bar{u}[$, solution of $\Gamma^\sigma(u;\tau) = 0$ with

$$\Gamma^\sigma(u;\tau) \equiv [(u-(1-\sigma))B - \sigma(\varrho+\lambda)] \times \\ \left\{\left[\alpha^{-1}-1+u\right]B + \mathcal{A}_0\Phi(\tau) - (\mathcal{A}_0+\eta)\,\lambda\right\} - \lambda\eta\,[(1-\sigma)B + \sigma(\lambda+\varrho)] \quad \text{(A.19)}$$

It is straightforward that $\Gamma^\sigma(\underline{\underline{u}};\tau) < 0$ (because the first term into brackets in the right-hand side of the equation is null) and $\Gamma(\bar{u};\tau)^\sigma > 0$. Because in the interval $]\underline{\underline{u}}, \bar{u}[$, $\Gamma^\sigma(u;\tau)$ is a monotonic increasing function of u, $u^\star$ solution of $\Gamma^\sigma(u;\tau) = 0$ is unique. Furthermore, from the implicit function theorem, $u^\star$ is a decreasing function of τ denoted $\mathcal{U}^\sigma(\tau) \equiv u^\star$, with $d\mathcal{U}^\sigma(\tau)/d\tau < 0$.

Because $g^{\star\sigma} = (1-\mathcal{U}^\sigma(\tau)) - (1-\eta)\lambda$, the growth rate along the BGP increases with τ.

Appendix E

This appendix gives the proof of propositions 6. In the way similar to Appendix B, $u^\star$ is the solution of $\Upsilon_\delta(u;\hat{\tau}) = 0$ where $\Upsilon(u;\hat{\tau})$ is defined as:

$$\Upsilon_\delta(u;\hat{\tau}) \equiv \left[(u-\delta)B\Delta^\delta\hat{\mathcal{R}}(B,\hat{\tau})^{-\alpha\delta} - \varrho - \lambda\right] \times \\ \left\{\left[u-\delta+(\alpha^{-1}-1)(1-\delta)\right]B\Delta^\delta\hat{\mathcal{R}}(B,\hat{\tau})^{-\alpha\delta} - (\alpha^{-1}-1+\eta)\lambda \right. \\ \left. + \frac{(1-u)(1-\alpha)\delta}{(1-\delta)u}\hat{\mathcal{R}}(B,\hat{\tau})^{1-\alpha}\right\} - \eta\lambda(\varrho+\lambda) \quad \text{(A.20)}$$

Because $x^\star > 0$ (the human capital accumulation can not exceed the maximum feasible rate of growth) and $g^\star > 0$, conditions (C1)-(C2) and the inequality (A.15) hold here: $u^\star \in]\underline{u}_\delta, \bar{u}_\delta[$.

In the way similar to Appendix B, we can demonstrate that $\Upsilon_\delta(\underline{u}_\delta;\hat{\tau}) = -\eta\lambda(\varrho+\lambda) < 0$ and $\Upsilon_\delta(\bar{u}_\delta;\hat{\tau}) > 0$. Because in the interval $]\underline{u}_\delta, \bar{u}_\delta[$, $\Upsilon_\delta(u;\hat{\tau})$ is a monotonic increasing function of u, $u^\star$ is solution of $\Upsilon(u;\hat{\tau}) = 0$ is unique.

When $\delta \neq 0$, the influence of $\hat{\tau}$ on $u^\star$ is very cumbersome to obtain analytically, therefore we just investigate the case $\delta = 0$.

When $\delta = 0$, $u^\star$ is the solution of $\Upsilon_\delta(u;\hat{\tau})\,|_{\delta=0} = 0$ where $\Upsilon_\delta(u;\hat{\tau})\,|_{\delta=0}$ is given by (A.20) with $\delta = 0$ and $\Delta^\delta\hat{\mathcal{R}}(B,\hat{\tau})^{-\alpha\delta} = 1$:

$$\Upsilon_\delta(u;\hat{\tau})\,|_{\delta=0} \equiv [uB - \varrho - \lambda] \times \left\{\left[u+\alpha^{-1}-1\right]B - (\alpha^{-1}-1+\eta)\lambda\right\} - \eta\lambda(\varrho+\lambda)$$

Because $\Upsilon_\delta(u;\hat{\tau})\,|_{\delta=0}$ is independent from $\hat{\tau}$, $u^\star$ is independent from $\hat{\tau}$. And $g^\star = B(1-u^\star) - (1-\eta)\lambda$ is independent from $\hat{\tau}$.

Appendix F

As noted by Calvo and Obstfeld (1988), the social welfare function, at time $t = 0$ is the sum of two components. The first component captures the expected utilities of agents from each of the generations to be born, measured from the moment of birth. The second component captures expected utilities of agents from each of those generations currently alive, over the remainder of their lifetimes, measured from the time $t = 0$. The planner discount rate is equal to the pure time-preference ϱ to avoid problems of time consistency (see Calvo and Obstfeld (1988) for more details). Consequently, welfare at $t = 0$ is

$$W_0 = \int_0^\infty \left\{ \int_s^\infty U[c(s,t), \mathcal{P}(t)] \lambda e^{-(\varrho+\lambda)(t-s)} dt \right\} e^{-\varrho s} ds + \int_{-\infty}^0 \left\{ \int_0^\infty U[c(s,t), \mathcal{P}(t)] \lambda e^{-(\varrho+\lambda)t+\lambda s} dt \right\} ds \quad \text{(A.21)}$$

Note that the second term in the right-hand side is discounted by the planner at time 0, so we write it as $\int_{-\infty}^0 \left\{ \int_0^\infty U[c(s,t), \mathcal{P}(t)] \lambda e^{-\lambda(t-s)} e^{-\varrho(t-0)} dt \right\} e^{-\varrho 0} ds$. After changing the order of derivation, we can write (A.21) as

$$W_0 = \int_0^\infty \left\{ \int_{-\infty}^t U[c(s,t), \mathcal{P}(t)] \lambda e^{-\lambda(t-s)} ds \right\} e^{-\varrho t} dt \quad \text{(A.22)}$$

The program of the social planner is:

$$\begin{aligned}
\max_{\substack{c(s,t),u(s,t),D(t)\\ K(t),H(t),h(s,t)}} \quad & \int_0^\infty \left\{ \int_{-\infty}^t [\log c(s,t) - \zeta \log \mathcal{P}(t)] \lambda e^{-\lambda(t-s)} ds \right\} e^{-\varrho t} dt \\
\text{s.t.} \quad & \dot{K}(t) = K(t)^\alpha [\textstyle\int_{-\infty}^t u(s,t) h(s,t) \lambda e^{-\lambda(t-s)} ds]^{1-\alpha} - \int_{-\infty}^t c(s,t) \lambda e^{-\lambda(t-s)} ds - D(t) \\
& \dot{H}(t) = \textstyle\int_{-\infty}^t \{B[1-u(s,t)] - (1-\eta)\lambda\} h(s,t) \lambda e^{-\lambda(t-s)} ds \\
& \mathcal{P}(t) = (K(t)/D(t))^\gamma \\
& H(t) = \textstyle\int_{-\infty}^t h(s,t) \lambda e^{-\lambda(t-s)} ds \\
& K(t) > 0, H(t) > 0, K_0 \text{ and } H_0 \text{ given,}
\end{aligned} \quad (24)$$

To solve (24), we define the Lagrangian:

$$\begin{aligned}
\mathcal{L} = {} & e^{-\varrho t} \left\{ \int_{-\infty}^t [\log c(s,t) - \zeta\gamma \log K(t) + \zeta\gamma \log D(t)] \lambda e^{-\lambda(t-s)} ds \right\} \\
& + \pi_1(t) \left\{ K(t)^\alpha \left[\int_{-\infty}^t u(s,t) h(s,t) \lambda e^{-\lambda(t-s)} ds \right]^{1-\alpha} - \int_{-\infty}^t c(s,t) \lambda e^{-\lambda(t-s)} ds - D(t) \right\} \\
& + \pi_2(t) \left\{ \int_{-\infty}^t \{B[1-u(s,t)] - (1-\eta)\lambda\} h(s,t) \lambda e^{-\lambda(t-s)} ds \right\} \\
& + v(t) \left\{ H(t) - \int_{-\infty}^t h(s,t) \lambda e^{-\lambda(t-s)} ds \right\}
\end{aligned}$$

where π_1 and π_2 are the costate variables for an interior solution and v is the Lagrangian multiplier.

The necessary conditions are:

$$\frac{\partial \mathcal{L}}{\partial c(s,t)} = 0 \quad \Rightarrow \quad e^{-\varrho t} c(s,t)^{-1} = \pi_1(t) \quad \text{(A.23)}$$

$$\frac{\partial \mathcal{L}}{\partial u(s,t)} = 0 \qquad \Rightarrow \qquad \pi_1(t)(1-\alpha)K(t)^{\alpha}\left[\int_{-\infty}^{t} u(s,t)h(s,t)\lambda e^{-\lambda(t-s)}ds\right]^{-\alpha} = \pi_2(t)B \quad \text{(A.24)}$$

$$\frac{\partial \mathcal{L}}{\partial D(t)} = 0 \qquad \Rightarrow \qquad \zeta\gamma e^{-\varrho t} = \pi_1(t)D(t) \quad \text{(A.25)}$$

$$\frac{\partial \mathcal{L}}{\partial K(t)} = -\dot{\pi}_1(t) \qquad \Rightarrow \qquad \zeta\gamma e^{-\varrho t}K(t)^{-1} + \pi_1(t)\alpha K(t)^{\alpha-1}\left[\int_{-\infty}^{t} u(s,t)h(s,t)\lambda e^{-\lambda(t-s)}ds\right]^{1-\alpha} = -\dot{\pi}_1(t) \quad \text{(A.26)}$$

$$\frac{\partial \mathcal{L}}{\partial H(t)} = -\dot{\pi}_2(t) \qquad \Rightarrow \qquad v(t) = -\dot{\pi}_2(t)$$

$$\frac{\partial \mathcal{L}}{\partial h(s,t)} = 0 \qquad \Rightarrow \qquad \pi_1(t)(1-\alpha)K(t)^{\alpha}\left[\int_{-\infty}^{t} u(s,t)h(s,t)\lambda e^{-\lambda(t-s)}ds\right]^{-\alpha} u(s,t) + \pi_2(t)\left\{B[1-u(s,t)] - \lambda(1-\eta)\right\} - v(t) = 0 \quad \text{(A.27)}$$

$$\lim_{t\to\infty} \pi_1(t)K(t)e^{-\varrho t} = 0 \qquad \text{and} \qquad \lim_{t\to\infty} \pi_2(t)H(t)e^{-\varrho t} = 0$$

First, (A.23) and (A.25) imply that $c(s,t)$ is independent from s: $c(s,t) = c(t)$. Consequently, because $C(t) = \int_{-\infty}^{t} c(s,t)\lambda e^{\lambda(t-s)}ds$, we have $c(s,t) = C(t)$. Based on (A.22), we write the social welfare function as

$$W_0 = \int_0^{\infty} U[C(t), \mathcal{P}(t)]e^{-\varrho t}dt.$$

Equation (A.27) means that $u(s,t)$ is also independent from s: $u(s,t) = u(t)$. From equations (A.24) and (A.27), we obtain

$$\frac{\dot{\pi_2(t)}}{\pi_2(t)} = \lambda(1-\eta) - B \quad \text{(A.28)}$$

Equations (A.25) and (A.26) give:

$$\frac{\dot{\pi_1(t)}}{\pi_1(t)} = \frac{D(t)}{K(t)} - \alpha K(t)^{\alpha-1}(u(t)H(t))^{1-\alpha} \quad \text{(A.29)}$$

Differentiating (A.24) with respect to time and using the previous results, it becomes:

$$\frac{\dot{u(t)}}{u(t)} = \alpha^{-1}\left(\frac{\dot{\pi_1(t)}}{\pi_1(t)} - \frac{\dot{\pi_2(t)}}{\pi_2(t)}\right) - \frac{\dot{H(t)}}{H(t)} + \frac{\dot{K(t)}}{K(t)} \quad \text{(A.30)}$$

Differentiating (A.23) with respect to time, we obtain:

$$\frac{\dot{C(t)}}{C(t)} = -\frac{\dot{\pi_1(t)}}{\pi_1(t)} - \varrho \quad \text{(A.31)}$$

Furthermore, from (A.23) and (A.25), we obtain $D(t) = \zeta\gamma C(t)$ and we have

$$\frac{\dot{K(t)}}{K(t)} = K(t)^{\alpha-1}(u(t)H(t))^{1-\alpha} - (1+\zeta\gamma)\frac{C(t)}{K(t)} \tag{A.32}$$

and

$$\frac{\dot{H(t)}}{H(t)} = B(1-u(t)) - (1-\eta)\lambda \tag{A.33}$$

Along the Balanced Growth Path, from (A.31), (A.32) and (A.29), $\dot{x} = 0$ implies

$$(\alpha - 1)(b_c^\star u_c^\star)^{1-\alpha} - \varrho + x_c^\star = 0 \tag{A.34}$$

and from (A.32), (A.33) and (A.29), $\dot{b} = 0$ implies

$$B(1-u_c^\star) - \lambda(1-\eta) = (b_c^\star u_c^\star)^{1-\alpha} - (1+\zeta\gamma)x_c^\star \tag{A.35}$$

where a star denotes a variable along the BGP. Furthermore, with $\dot{u} = 0$, (A.29), (A.28) and (A.30) give

$$\alpha(b_c^\star u_c^\star)^{1-\alpha} - \zeta\gamma x_c^\star = B - \lambda(1-\eta) \tag{A.36}$$

that is the return to the accumulation of physical capital equal the return to accumulation of human capital. Equations (A.35) and (A.36) give:

$$x_c^\star = \frac{(1-\alpha)\left[B - (1-\eta)\lambda\right] + \alpha\varrho}{\alpha - (1-\alpha)\zeta\gamma} \tag{A.37}$$

$x_c^\star$ is positif because $\dot{K} > 0$ requires that $\zeta\gamma < \frac{(bu)^{1-\alpha}}{x} - 1$ from equation (A.32). And using (A.34), it implies that $(\alpha - (1-\alpha)\zeta\gamma)\, x_c^\star > \varrho$. Replacing by the expression of $x_c^\star$ found in equation (A.37), we obtain

$$B - (1-\eta)\lambda - \varrho > 0 \tag{A.38}$$

Subtracting (A.34) and (A.35) and using (A.36), we obtain the value of the allocation of human capital to production in the long-run $u_c^\star = \varrho/B \in]0,1[$, from (A.38). Finally, using the value of $u_c^\star$, the BGP rate of growth in the centralized economy, $g_c^\star = B - \varrho - \lambda(1-\eta) > 0$, does not depend on the long-run flow of pollution and on the environmental care ζ.

It remains to calculate the optimal environmental tax, denoted ϑ^{op}. To obtain the expression of this tax, we equalize the expression of the pollution in the market economy (equation 13) with the expression of pollution in the centralized economy given by the ratio $(K/D)^\gamma$. From (A.23), (A.25) and (A.37), we obtain $\vartheta^{op}(t) = K(t)\frac{(\zeta\gamma)^{1+\gamma}}{\gamma}x(t)^{1+\gamma}$, that is, denoted τ^{op} the optimal tax normalized by the physical capital stock:

$$\tau^{op} = \frac{(\zeta\gamma)^{1+\gamma}}{\gamma}x(t)^{1+\gamma}$$

When the government chooses the environmental tax equal to

$$\tau^{op}|_{BGP} = \frac{(\zeta\gamma)^{1+\gamma}}{\gamma}\left(\frac{(1-\alpha)\left[B-(1-\eta)\lambda\right]+\alpha\varrho}{\alpha-(1-\alpha)\zeta\gamma}\right)^{1+\gamma}$$

along the BGP, pollution is at its optimal level.